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Chen Jingrun: A Brief Outline of His Life and Works

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Chen Jingrun was born on 22 May, 1933 in Fu Zhou in Fu Jian province of China. Chen's father Chen Yuanjun was a clerk in postoffice and his mother was passed away in 1947. Chen's family was comparatively poor since the income of his father was lower and the population of his family was comparatively large.

After the graduation of Chen in a middle school in Fu Zhou, He entered the department of Mathematics, Xia Men University in 1949. Chen was appointed by the government as a teacher of the Beijing Fourth middle school when he was graduated in 1953. He was fired by the school because he was not suited for his job. Mr. Wang Yanan, the president of Xia Men University, was informed on Chen's situation, and then he introduced Chen as a clerk in Xia Men University in 1955.

Chen was very interested in number theory in that time. Xia Men was a city in the coastal front. The air raid alarms were often appeared and the people should hid in the air raid shelter. Chen brought several pages of Hua Loogeng's book "Additive Prime Number Theory" and studied it even in the shelter. The Chapter 4 "Mean value theorems of certain trigonometric sums II" in Hua's book is treated by Hua's method the mean value theorems of trigonometric sum of polynomial with lower degrees, and the Chapter 5 "Vinogradov's mean value theorem and its applications" is devoted to the mean value theorems of trigonometric sum of polynomial with higher degrees by Vinogradov's method. Chen succeded to use the method in Chapter 5 to improve some results in Chapter 4 of Hua's book. He wrote a paper "On Tarry's problem" and mailed to Hua. Hua was confidence in that Chen has high talent in mathematics after Chen's paper was confirmed by some mathematicians in the number theory section of the Institute of Mathematics, Academia Sinica.

Qwing to Hua's introduction, Chen attended the Annual meeting of the Chinese mathematical Soliety held in August, 1956, and gave a lecture on his result of Tarry's problem. The participants were interested in Chen's talk. Then Chen became an assistant in the Institute of Mathematics, Academia Sinica in 1957 when he was highly recommended by Hua.

Chen's research works have important progresses when he stayed in the Institute. Using the estimation of trigonometric sums and their applications, he pushed forward some records of the famous circle problem, divisor problem, sphere problem and E. Waring's problem.

Chen attended the climax of his research when he studied the sieve method and its applications in the 60's. His results on C. Goldbach conjecture and the distribution of almost primes have wide international influences and high appreciations.

Chen was often sick and his healthy was not well. The erroneous criticisms and serious attacks were often imposed on Chen in the so-called "Cultural Revolution" between 1966 and

1976 so that Chen's works and healthy were seriously injured. Chen was unfortunately to suffer from the Parkinson's decease in 1984. He still continued his works and discussed with some young mathematicians even in this situation. Chen's life and work have good care by the government when Cultural Revolution was ceased, and he got even good care when he stayed in the hospital and his sick became serious. Chen was passed away in March 19, 1996.

Owing to the important contuibutions in mathematics, Chen was appointed as the research professor of the Institute in 1978 and elected as the member of Academia Sinica in 1980. Chen was awarded the 1st rank of National Natural Science Prize, He-Liang-He-Li Prize and Hua Lookeng Mathematics Prize.

Chen was married with You Kun in 1980, and has a son Chen Youwei.

Mathematical Works

A. Sieve Methods and their applications

1 Representation of large even integer as the sum of a prime and an almost prime

In a letter to L. Euler in 1742, Goldbach proposed two conjectures on the representation of integers as the sum of primes:

(A) Every even integer ≥ 6 is the sum of two odd primes.

(B) Every odd integer ≥ 9 can be represented as the sum of three odd primes.

Evidently, we can derive (B) from (A). I. M. Vinogradov proved in 1937 the conjecture (B) for large odd integers based on the circle method and his ingeneous estimation of trigonometric sums with prime variables. Therefore it remarins to prove the conjecture (A) only. We can prove also by the Vinogradov's method that almost all even integers are sums of two primes. More precisely, let $E(x)$ denote the number of even integers $\leq x$ which cannot be represented as sums of two primes. Then $E(x) = O(x(\ln x)^{-B})$, where B is any given positive number and the constant implicited in O depends on B.

The sieve method is another way to treat the conjecture (A). The historical origin of sieve method may be traced back to the "sieve of Eratosthenes" about 250 B.C. It was a great achievement when V. Brun in 1919 devised his new sieve method and applied to conjecture (A). Let P_a be an integer satisfying the following condition: the number of prime factors of P_a is at most a. P_a is called to be an almost prime. Brun proved the following result:

(1) Every large even integer is a sum of two almost primes P_9 and Q_9. For simplicity, we denote this result by $(9,9)$.

We can define similarly (a,b). Brun's method and his result were improved by several mathematiciens, namely $(7,7)$ (H. Rademacher, 1924), $(6,6)$ (T. Estermann, 1932). $(5,5)$ (A. A. Buchstab, 1938), $(4,4)$ (A. A. Buchstab, 1940) and (a,b) ($a+b \leq 6$, P. Kulen, 1954). The power of Brun's method will be vastly improved if some combinatorial relations are used, and these combinatorial ideas were introduced by Buchstab and Kuhn. Another important improvement of Eratosthenes sieve was given by A. Selberg in 1947. By the combination of all methods mentioned above. Wang Yuan proved $(3,4)$ (1956) and $(2,3)$ (1957). By the use of Brun's method, the theory of distribution of primes and the large sieve of Yu. V. Linnik, A. Renyi established in 1948 the following

(2) Every large even number is a sum of a prime and an almost prime with at most c prime

factors, where c is constant, i.e. $(1, c)$.

Let $\pi(x;k,l)$ be the number of primes satisfying $p \le x, p \equiv l \pmod{k}$. The key step in Renyi's proof of $(1, c)$, is that a mean value theorem for $\pi(x;k,l)$ is proved: There exists a positive number $\delta > 0$ such that

$$\sum_{k \le x^\delta} \max_{(l,k)=1} \left| \pi(x;k,l) - \frac{\operatorname{Li} x}{\varphi(k)} \right| = O\left(\frac{x}{(\ln x)^{c_1}} \right), \tag{3}$$

where $\varphi(k)$ is the Euler's function, $\operatorname{Li} x = \int_2^x \frac{dt}{\ln t}$ and c_1 is a constant ≥ 5. In 1961 and 1962. M. B. Barban and Pan Chengtong proved independently that (3) are ture for $\delta = \frac{1}{6} - \varepsilon$ and $\delta = \frac{1}{3} - \varepsilon$ respectively, where ε is any positive and the constant implicited in O in (3) depends on ε. Pan gave (1.5) as an application of his $\delta = \frac{1}{3} - \varepsilon$. In 1962 and 1963. Pan and Barban improved (3) to $\delta = \frac{3}{8} - \varepsilon$ and derived (1,4). Notice that sometimes $\pi(x;k,l)$ in (3) should be replaced by a weight sum. In 1965, A. I. Vinogradov and E. Bombiei proved independently (3) with $\delta = \frac{1}{2} - \varepsilon$, and so it follows (1,3). More precisely, the range of k in Bombieri's result is $x^{1/2}/(\ln x)^{c_2}$, where c_2 is a constant depending on c_1. The importance of Bombieri-A.Vinogradov's formula is that it can be used sometimes instead of Grand Riemann Hypothesis. In 1966,Chen introduced ingeneously a switching principle, and proved (1,2):

(4) Every large even integer is a sum of a prime and an almost prime with at most 2 prime factors.

Let us explain his work in more details.

Let p, p_1, p_2, p_3 denote primes, $A = \{a_v\}$ a finite set of integers and $F(A;q,q')$ the number of elements in A satisfying

$$a_v \equiv 0 (\bmod\ q), \quad a_v \not\equiv 0 (\bmod\ p) \qquad (p < q', p \nmid q').$$

In particular we denote by $F(A;q') = F(A;1,q')$. Let n be an even number, $A = \{n-p, p < n\}$,

$$N = F(A; n^{\frac{1}{10}}) - \frac{1}{2} \sum_{n^{1/10} \le p < n^{1/3}} F(A;p,p^{1/10}),$$

$$\Omega = \frac{1}{2} \sum_{\substack{p<n \\ (p_{1,2})}} \sum_{\substack{n-p=p_1p_2p_3 \\ p_3 \le n/p_1p_2}} 1 \quad \text{and} \quad M = N - \Omega + O(n^{9/10}),$$

where $(p_{1,2})$ denotes the condition $n^{1/10} \le p_1 < n^{1/3} \le p_2 \le \left(\frac{n}{p_1}\right)^{1/2}$. We can obtain a positive lower estimation for N by the use of Bombieri-A. Vinogradov's mean value theorem and various sieve methods, and it yields (1,3). Chen introduced Ω and gave it an upper estimation such that M has a positive lower estimation, so he proved (1,2). ([8,9]).

2 Estimation of the number of solutions for even integer as a sum of two primes

Let n be an even integer and $D(n) = \sum_{p_1+p_2=n} 1$ the number of representations of n as a sum of two primes. Applying Selberg's sieve method to the set $A = \{a_v = v(n-v), 1 \le v < n\}$ we obtain

$$D(n) \le 16\sigma(n) \frac{n}{(\ln n)^2}(1 + o(1))$$

where

$$\sigma(n)=\prod_{p|n}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

If we use Bombieri -A. Vinogradov's mean value theorem and applying Selberg's sieve method to the set $A=\{a_p=n-p,p<n\}$, then we have

$$D(n)\le 8\sigma(n)\frac{n}{(\ln n)^2}(1+o(1)).$$

It is difficult to improve the coefficient 8 in above formula. Chen improved the 8 by 7.8342 in 1978, i.e.;

$$D(n)\le 7.8342\sigma(n)\frac{n}{(\ln n)^2}(1+o(1))\qquad ([14]). \tag{5}$$

3 Distribution of almost primes

There is a famous conjecture in prime number theory:

(C) There exists always a prime in the interval $[x,x+2x^{1/2}]$ when $x\ge 1$.

Burn was the first who proved by sieve method that there is an almost prime P_{11} in the interval $[x,x+x^{1/2}]$ when x is sufficiently large. Brun's result was improved by several mathematicians, for example, Wang established that $P_3\in[x,x+x^{20/49}](x>x_0)$ in 1957. We are interested the following problem: Is conjective (C) true when p is replaced by P_2? Wang proved in 1957 that there is a P_2 such that

$$P_2\in[x,x+x^{\frac{10}{17}}],\qquad (x>x_0).$$

H. E. Richent improved the above result to

$$P_2\in[x,x+x^{\frac{6}{11}}],\qquad (x>x_0)$$

in 1969, Chen proved conjecture (C) for P_2 in 1975, that is, there is a P_2 such that

$$P_2\in[x,x+x^{1/2}]\qquad (x>x_0). \tag{6}$$

In the proof of (6), Chen used the weight sieve method, and the estimation of trigonometric sum was firstly used by him to estimate the error term appeared in the sieve method. In 1979, Chen Jingrun improved the $x^{1/2}$ in (6) to $x^{0.4777}$ by the combinatorial idea. Chen's method is the starting point in many later important works. ([11, 16]).

B. Other works

4 Waring's problem

The waring's problem was proposed by British mathematician Waring in 1770 on the representation of positive integer as sum of same positive integer powers:

(D) For any given integer $k\ge 2$, there exists an integer $s=s(k)$ depending on k such that every positive integer is a sum of s k-th power of nonnegative integers.

This historical problem was solved by D. Hilbert in 1908. Let $g(k)$ be the least integer of $s=s(k)$ such that (D) holds for all positive integers. We are interested to ask that what is

$g(k)$ or the estimation of $g(k)$. It is known that $g(2) = 4$ (Euler and J. L. Lagrange, 1770) and $g(3) = 9$ (A. Wieferich). L. E. Dickson and S. S. Pillai established independently that if $k > 6$ and

$$\left(\frac{3}{2}\right)^k - \left[\left(\frac{3}{2}\right)^k\right] \le 1 - \left(\frac{1}{2}\right)^k \left\{\left[\left(\frac{3}{2}\right)^k\right] + 3\right\}, \tag{7}$$

then

$$g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2,$$

where $[x]$ denotes the integral part of x. Pillai proved also that $g(6) = 73$. Therefore it remains only to treat the cases of $k = 4, 5$ and the k such that (7) does not hold. Chen solved the case $k = 5$ in 1964, i.e.,

$$g(5) = 37. \tag{8}$$

We can derive also $g(4) \le 20$ by Chen's method. ([1,5,10]). Until 1986, R. Balasubramanian, J. M. Deshouiller and F. Dress proved the case $g(4) = 19$. ([48]).

5 Lattice points problems

Let $r(n)$ be the number of solutions of positive integer n as a sum of two integer squares and $r(0) = 1$. Then

$$A(x) = \sum_{0 \le n \le x} r(n)$$

is the number of lattice points (u, v) in the circle $u^2 + v^2 \le x$. Let $d(n)$ denote the number of divisors of positive integer n. Then

$$D(x) = \sum_{1 \le n \le x} d(n)$$

is the number of lattice points in the domain $uv \le x, u \ge 1, v \ge 1$. The so-called circle problem and divisor problem are to find respectively the least θ and φ such that

$$A(x) = \pi x + O(x^{\theta+\varepsilon}) \quad \text{and} \quad D(x) = x(\ln x + 2\gamma - 1) + O(x^{\varphi+\varepsilon})$$

hold for any given positive ε, where γ denotes the Euler constant and the constants implicited in O depending on ε. There is a famous conjecture in number theory:

$$\theta = \varphi = \frac{1}{4}. \tag{D}$$

Another famous problem is for finding the order of magnititude of Riemann ζ-function on the critical line. Since the methods for treating these problems are involved the similar trigonometric sums, we state only the progresses of circle problem. C.F. Gauss proved first that $\theta = \frac{1}{2}$. G. Voronoi gave a great improvement in 1903, and proved that $\varphi = \frac{1}{3}$. W. Sierpinski established also in 1906 that $\theta = \frac{1}{3}$. J. G. Van der corput introduced in 1923 the estimation of certain trigonometrical sums so that he proved $\theta = \frac{37}{112}$. Up to 1942, the best record $\theta = \frac{13}{40}$ is due to Hua. Chen improved the result to $\frac{12}{37}$, i.e.,

$$A(x) = \pi x + O(x^{\frac{12}{37}+\varepsilon}). \quad ([43]) \tag{9}$$

Now the best record $\theta = \frac{7}{22}$ is due to H. Iwaniec and J. Mozzochi ([49]).

There are so called sphere problem and problem of the averge of class numbers of imaginary quadratic fields similar to circle problem and divisor problem. More precisely. Let $B(x)$ be the number of lattice points (u, v, w) in the sphere $u^2 + v^2 + w^2 \leq x$. The sphere problem is to find the least θ_1 such that for any given $\varepsilon > 0$.

$$B(x) = \frac{4}{3}\pi x^{3/2} + O(x^{\theta_1+\varepsilon}).$$

Let d be an integer > 0 and $h(-d)$ the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. The problem of average of class numbers is to find the least φ_1 such that for any given $\varepsilon > 0$,

$$H(x) = \sum_{1 \leq d \leq x} h(-d) = \frac{4\pi}{21\zeta(3)} x^{3/2} - \frac{2}{\pi^2} x + O(x^{\varphi_1+\varepsilon}).$$

Chen and Vinogradov proved independently in 1963 the following

$$\theta_1 = \varphi_1 = 2/3. \tag{10}$$

θ_1 and φ_1 have the improvements similar to circle problem.

6 Least prime in an arithmetic progression

Let k, l be positive integers satisfyingg $(k, l) = 1$. Are there infinitely many primes in the arithmetic progression

$$kn + l, \quad n = 0, 1, 2, \ldots?$$

This problem was solved by D. G. L. Dirichlet in 1837. Let $P(k, l)$ denote the least prime in the above arithmetic progression. S.Chowla proposed the conjecture in 1934 that for any $\varepsilon > 0$.

$$P(k, l) = O(k^{1+\varepsilon}), \tag{E}$$

where the constant implicited in O depends on ε. It was Linnik who proved that there exists a constant c such that

$$P(k, l) = O(k^c).$$

Pan gave first an estimation $c \leq 5448$ in 1957. Later many mathematicians improved the Pan's result, where Chen and his students gave the following records of c:

$$777, 168, 17, 15, 13.5, 11.5 \quad ([7, 12, 15, 27, 29, 30, 31]) \tag{12}$$

Now the best record is due to D. R. Heath-Brown ([50]).

7 Goldbach numbers

The even number which can be represented as a sum of two primes is called a Goldbach number. The $E(x)$ defined in §1 is the number of even integers $\leq x$ which are not the Goldbach numbers. H. L. Montgomery and R. C. Vaughan improved the estimation of $E(x)$, and they proved that there exists $\delta > 0$ such that

$$E(x) = O(x^{1-\delta}),$$

where the constant implicited in O depends on δ ([51]). Chen and Pan determined that

$$\delta > 0.01. \tag{13}$$

Later, Chen Jing Run improved the estimation of δ to $\delta > 0.05$ ([17, 18, 19, 35]).

8 Estimation of Trigonometric sums

Let q be an integer ≥ 2 and $f(x) = a_k x^k + \cdots + a_1 x$ a k-th polynomial with integral coefficients such that $(a_{k_1}, \ldots, a_1, q) = 1$. Let

$$S(f(x), q) = \sum_{x=1}^{q} e(f(x)/q),$$

where $e(y) = e^{2\pi i y}$. This is called to be complete exponential sum. When $f(x) = ax^2, S(ax^2, q)$ is the famous Gaussian sum, and Gauss gave an estimation

$$|S(ax^2, q)| \leq 2\sqrt{q}. \tag{14}$$

The famous problem of estimation of the general complete exponential sum, was solved by Hua in 1940. In fact, he proved proved by A. Weil's theorem that

$$|S(f(x), q)| \leq c(k) q^{1-\frac{1}{k}}, \tag{15}$$

where the order $1 - \dfrac{1}{k}$ in the right side of (15) is of best posible and where the constant $c(k)$ depends on k. Chen gave the estimation of $c(k)$:

$$c(k) = \begin{cases} \exp(4k), & \text{if } k \geq 10, \\ \exp(kA(k)), & \text{if } 3 \leq k \leq 9, \end{cases} \tag{16}$$

where $\exp(x) = e^x$ and $A(3) = 6.1, A(4) = 5.5, A(5) = 5, A(6) = 4.7, A(7) = 4.4, A(8) = 4.2, A(9) = 4.05$ ([20]).

Acknowledgement: There are many publications concerning Chen's life and workds, for examples [41, 42, 43, 44, 45, 46, 47]. A part of material of this paper is taken from these publications.

A List of Chen Jingrun's Papers

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