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# 锥 b- 度量空间中的三元重合点 与三元不动点定理

罗 婷

江西财经大学信息管理学院 南昌 330032  
E-mail: xiaoxiaoluo\_66@163.com

朱传喜

南昌大学数学系 南昌 330031  
E-mail: chuanxizhu@126.com

**摘 要** 本文在锥 b- 度量空间中引进了映射对  $F : X \times X \times X \rightarrow X$  与  $g : X \rightarrow X$  的三元重合点与弱相容性的新概念. 在锥不需要正规性的条件下, 研究了压缩映射对的三元重合点与具有弱相容性映射对的三元公共不动点问题, 所得结果推广了已有文献中的二元重合点与二元公共不动点定理. 最后给出主要结果的一个应用.

**关键词** 锥 b- 度量空间; 三元重合点; 三元不动点; 半序集; 弱相容映射

**MR(2010) 主题分类** 47H10

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## Tripled Coincidence Point and Tripled Fixed Point Theorems in cone b-metric Spaces

Ting LUO

*School of Information Management, Jiangxi University of Finance and Economics,  
Nanchang 330032, P. R. China  
E-mail: xiaoxiaoluo\_66@163.com*

Chuan Xi ZHU

*Department of Mathematics, Nanchang University, Nanchang 330031, P. R. China  
E-mail: chuanxizhu@126.com*

**Abstract** In this paper, the new concept of tripled coincidence point and weakly compatible for a pair of mappings  $F : X \times X \times X \rightarrow X$ ,  $g : X \rightarrow X$  in cone b-metric spaces are introduced. Under not necessary normal conditions of cone, some tripled coincidence for contractive mappings and tripled common fixed point problems of weakly compatible mappings are studied. The obtained results generalize some coupled common fixed point theorems in corresponding literatures. Finally, an example is given to illustrate our main results.

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**Keywords** cone b-metric space; tripled coincidence point; tripled fixed point; partially ordered set; weakly compatible mapping

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## 1 引言

Bakhtin<sup>[2]</sup> 在 1989 年提出了 b-度量空间的概念,它是度量空间的一个推广. 有关在 b-度量空间中研究不动点问题的文章见 [3–5, 11]. 随后, Huang 和 Zhang<sup>[8]</sup> 引入了锥度量空间,并定义了其上的拓扑. 有关锥度量空间中压缩映射的不动点的研究见文 [1, 10, 14, 16, 17]. 在 2011 年, Hussain 和 Shah<sup>[9]</sup> 引入了锥 b-度量空间的概念,推广了 b-度量空间与锥度量空间. 有关锥 b-度量空间中压缩映射的不动点的研究见文 [7, 9, 12, 13, 15]. 最近, Fadail 和 Bin Ahmad<sup>[6]</sup> 在锥 b-度量空间中研究了映射对的二元重合点与二元公共不动点. 本文在锥 b-度量空间中,在不要求锥具有正规性的条件下,研究了映射对  $F: X \times X \times X \rightarrow X$  与  $g: X \rightarrow X$  的三元重合点问题. 然后通过定义  $F$  与  $g$  的弱相容性概念,研究了  $F$  与  $g$  的三元公共不动点问题. 所得结果推广了已有文献中的二元重合点与二元公共不动点定理. 最后,给出主要结果的一个应用.

先回顾如下概念.

设  $\mathbb{R}$  表示全体实数所成集合,  $\mathbb{R}^+$  表示全体非负实数所成集合,  $\mathbb{N}$  表示全体自然数所成集合.

设  $E$  是实 Banach 空间,  $P$  是  $E$  中的非空凸闭集. 如果  $P$  满足:

(i)  $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$ ;

(ii)  $x \in P, -x \in P \Rightarrow x = \theta$ ,

则称  $P$  是  $E$  中一个锥.

给定 Banach 空间  $E$  中一个锥  $P$  后,可以在  $E$  中引入半序关系如下:  $x \preceq y$ , 如果  $y - x \in P$ . 如果  $x \preceq y$  并且  $x \neq y$ , 则我们记  $x < y$ ; 如果  $P$  是一个体锥, 并且  $y - x \in \text{int } P$ ,  $\text{int } P$  表示  $P$  的内部, 则我们记  $x \ll y$ . 如果存在常数  $N > 0$ , 使得

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|,$$

则称  $P$  是正规的. 满足上面条件的正数  $N$  中的最小者叫做  $P$  的正规常数. 本文中, 不需要锥的正规性, 只需要锥  $P$  是一个体锥, 即  $\text{int } P \neq \phi$ .

**定义 1.1**<sup>[9]</sup> 设  $X$  是一个非空集,  $E$  是实 Banach 空间,  $P$  是  $E$  中的锥,  $\preceq$  由  $P$  导出. 向量值函数  $d: X \times X \rightarrow E$  称为  $X$  中的锥 b-度量,  $s \geq 1$ , 如果下面条件成立:

(1)  $\theta \preceq d(x, y), \forall x, y \in X; d(x, y) = \theta \Leftrightarrow x = y$ ;

(2)  $d(x, y) = d(y, x), \forall x, y \in X$ ;

(3)  $d(x, z) \preceq s(d(x, y) + d(y, z)), \forall x, y, z \in X$ ,

则称  $(X, d)$  为锥 b-度量空间.

如果  $s = 1$ , 那么锥度量空间中的一般三角不等式成立; 然而, 当  $s > 1$  时并不成立. 因而, 锥 b-度量空间推广了锥度量空间. 也就是说, 锥度量空间是锥 b-度量空间, 但反过来不成立.

**例 1.2**<sup>[12]</sup> 设  $X = \{-1, 0, 1\}$ ,  $E = \mathbb{R}^2$ ,  $P = \{(x, y) : x \geq 0, y \geq 0\}$ . 定义  $d: X \times X \rightarrow P$  为  $d(x, y) = d(y, x), \forall x, y \in X, d(x, y) = \theta, x \in X$ , 并且  $d(-1, 0) = (3, 3), d(-1, 1) =$

$d(0,1) = (1,1)$ . 那么  $(X,d)$  是一个完备的锥 b- 度量空间, 但是三角不等式不成立. 事实上,  $d(-1,1) + d(1,0) = (1,1) + (1,1) = (2,2) \prec (3,3) = d(-1,0)$ .

**定义 1.3** <sup>[9]</sup> 设  $(X,d)$  是一个锥 b- 度量空间,  $\{x_n\}$  是  $X$  中的序列,  $x \in X$ .

(1)  $\{x_n\}$  称为收敛于  $x \in X$ , 如果对任意的  $c \in E$ ,  $\theta \ll c$ , 存在正数  $N$  使得  $d(x_n, x) \ll c$ ,  $\forall n > N$ . 记为  $x_n \rightarrow x$ .

(2)  $\{x_n\}$  称为  $X$  中的 Cauchy 列, 如果对任意的  $c \in E$ ,  $\theta \ll c$ , 存在正数  $N$  使得  $d(x_n, x_m) \ll c$ ,  $\forall n, m > N$ .

(3) 锥 b- 度量空间  $(X,d)$  称为完备的, 如果  $X$  中的 Cauchy 列是收敛的.

**引理 1.4** <sup>[10]</sup> (1)  $E$  是一个实 Banach 空间,  $P$  是  $E$  中的锥, 如果  $a \preceq \lambda a$ ,  $a \in P$ ,  $0 \leq \lambda < 1$ , 则  $a = \theta$ .

(2) 如果  $c \in \text{int } P$ ,  $\theta \preceq a_n$ ,  $a_n \rightarrow \theta$ , 则存在一个正数  $N$ , 使得  $a_n \ll c$ ,  $\forall n \geq N$ .

(3) 如果  $a \preceq b$ ,  $b \preceq c$ , 则  $a \preceq c$ .

(4) 如果  $\theta \preceq u \ll c$ ,  $\forall \theta \ll c$ , 则  $u = \theta$ .

## 2 锥 b- 度量空间中的三元重合点

**定义 2.1** 元素  $(x,y,z) \in X^3$  称为  $F: X \times X \times X \rightarrow X$  的三元不动点, 如果

$$F(x,y,z) = x, \quad F(y,z,x) = y, \quad F(z,x,y) = z.$$

**定义 2.2** 元素  $(x,y,z) \in X^3$  称为  $F: X \times X \times X \rightarrow X$  与  $g: X \rightarrow X$  的三元重合点, 如果

$$F(x,y,z) = gx, \quad F(y,z,x) = gy, \quad F(z,x,y) = gz.$$

**定义 2.3** 元素  $(x,y,z) \in X^3$  称为  $F: X \times X \times X \rightarrow X$  与  $g: X \rightarrow X$  的公共三元不动点, 如果

$$x = gx = F(x,y,z), \quad y = gy = F(y,z,x), \quad z = gz = F(z,x,y).$$

**定义 2.4** 设  $X$  为非空集,  $F: X \times X \times X \rightarrow X$  与  $g: X \rightarrow X$  称为弱相容的, 如果  $gx = F(x,y,z)$ ,  $gy = F(y,z,x)$ ,  $gz = F(z,x,y)$ , 有  $g(F(x,y,z)) = F(gx,gy,gz)$ .

**定理 2.5** 设  $(X,d)$  是锥 b- 度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F: X \times X \times X \rightarrow X$ ,  $g: X \rightarrow X$ , 假设存在非负常数  $a_i \in [0,1)$ ,  $i = 1, 2, \dots, 15$ ,  $(s+1)(a_1+a_2+a_3+a_4+a_5+a_6) + s(s+1)(a_7+a_8+a_9+a_{10}+a_{11}+a_{12}) + 2s(a_{13}+a_{14}+a_{15}) < 2$ , 且  $\sum_{i=1}^{15} a_i < 1$ , 使得对任意的  $x,y,z,u,v,w \in X$ , 有

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \preceq & [a_1 d(gx, F(x,y,z)) + a_2 d(gy, F(y,z,x)) + a_3 d(gz, F(z,x,y))] \\ & + [a_4 d(gu, F(u,v,w)) + a_5 d(gv, F(v,w,u)) + a_6 d(gw, F(w,u,v))] \\ & + [a_7 d(gx, F(u,v,w)) + a_8 d(gy, F(v,w,u)) + a_9 d(gz, F(w,u,v))] \\ & + [a_{10} d(gu, F(x,y,z)) + a_{11} d(gv, F(y,z,x)) + a_{12} d(gw, F(z,x,y))] \\ & + [a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw)]. \end{aligned} \quad (2.1)$$

若  $F(X^3) \subseteq g(X)$ , 并且  $g(X)$  是  $X$  中完备子空间, 则  $F$  和  $g$  有三元重合点  $(x^*, y^*, z^*) \in X^3$ .

**证明** 对任意的  $x_0, y_0, z_0 \in X$ , 因为  $F(X^3) \subseteq g(X)$ , 故存在  $x_1, y_1, z_1 \in X$ , 使得

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, z_0, x_0), \quad gz_1 = F(z_0, x_0, y_0).$$

依此类推, 存在  $\{x_n\}, \{y_n\}, \{z_n\} \subseteq X$ , 使得对任意  $n \geq 0$ , 有

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, z_n, x_n), \quad gz_{n+1} = F(z_n, x_n, y_n). \quad (2.2)$$

在 (2.1) 式中, 令  $x = x_{n-1}, y = y_{n-1}, z = z_{n-1}, u = x_n, v = y_n, w = z_n$ , 有

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ &\preceq [a_1 d(gx_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})) + a_2 d(gy_{n-1}, F(y_{n-1}, z_{n-1}, x_{n-1})) \\ &\quad + a_3 d(gz_{n-1}, F(z_{n-1}, x_{n-1}, y_{n-1}))] \\ &\quad + [a_4 d(gx_n, F(x_n, y_n, z_n)) + a_5 d(gy_n, F(y_n, z_n, x_n)) + a_6 d(gz_n, F(z_n, x_n, y_n))] \\ &\quad + [a_7 d(gx_{n-1}, F(x_n, y_n, z_n)) + a_8 d(gy_{n-1}, F(y_n, z_n, x_n)) + a_9 d(gz_{n-1}, F(z_n, x_n, y_n))] \\ &\quad + [a_{10} d(gx_n, F(x_{n-1}, y_{n-1}, z_{n-1})) + a_{11} d(gy_n, F(y_{n-1}, z_{n-1}, x_{n-1})) \\ &\quad + a_{12} d(gz_n, F(z_{n-1}, x_{n-1}, y_{n-1}))] \\ &\quad + [a_{13} d(gx_{n-1}, gx_n) + a_{14} d(gy_{n-1}, gy_n) + a_{15} d(gz_{n-1}, gz_n)], \end{aligned}$$

由 (2.2) 式可得

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ &\preceq [a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) + a_3 d(gz_{n-1}, gz_n)] \\ &\quad + [a_4 d(gx_n, gx_{n+1}) + a_5 d(gy_n, gy_{n+1}) + a_6 d(gz_n, gz_{n+1})] \\ &\quad + [a_7 d(gx_{n-1}, gx_{n+1}) + a_8 d(gy_{n-1}, gy_{n+1}) + a_9 d(gz_{n-1}, gz_{n+1})] \\ &\quad + [a_{10} d(gx_n, gx_n) + a_{11} d(gy_n, gy_n) + a_{12} d(gz_n, gz_n)] \\ &\quad + [[a_{13} d(gx_{n-1}, gx_n) + a_{14} d(gy_{n-1}, gy_n) + a_{15} d(gz_{n-1}, gz_n)] \\ &\preceq [a_1 d(gx_{n-1}, gx_n) + a_2 d(gy_{n-1}, gy_n) + a_3 d(gz_{n-1}, gz_n)] \\ &\quad + [a_4 d(gx_n, gx_{n+1}) + a_5 d(gy_n, gy_{n+1}) + a_6 d(gz_n, gz_{n+1})] \\ &\quad + [sa_7 (d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})) + sa_8 (d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1})) \\ &\quad + sa_9 (d(gz_{n-1}, gz_n) + d(gz_n, gz_{n+1}))] \\ &\quad + [a_{13} d(gx_{n-1}, gx_n) + a_{14} d(gy_{n-1}, gy_n) + a_{15} d(gz_{n-1}, gz_n)], \end{aligned}$$

从而有

$$\begin{aligned} d(gx_n, gx_{n+1}) &\preceq [(a_1 + sa_7 + a_{13})d(gx_{n-1}, gx_n) + (a_2 + sa_8 + a_{14})d(gy_{n-1}, gy_n) \\ &\quad + (a_3 + sa_9 + a_{15})d(gz_{n-1}, gz_n)] \\ &\quad + [(a_4 + sa_7)d(gx_n, gx_{n+1}) + (a_5 + sa_8)d(gy_n, gy_{n+1}) + (a_6 + sa_9)d(gz_n, gz_{n+1})], \quad (2.3) \end{aligned}$$

同理可得

$$\begin{aligned} d(gy_n, gy_{n+1}) &\preceq [(a_1 + sa_7 + a_{13})d(gy_{n-1}, gy_n) + (a_2 + sa_8 + a_{14})d(gz_{n-1}, gz_n) \\ &\quad + (a_3 + sa_9 + a_{15})d(gx_{n-1}, gx_n)] \\ &\quad + [(a_4 + sa_7)d(gy_n, gy_{n+1}) + (a_5 + sa_8)d(gz_n, gz_{n+1}) \\ &\quad + (a_6 + sa_9)d(gx_n, gx_{n+1})], \quad (2.4) \end{aligned}$$

$$\begin{aligned}
d(gz_n, gz_{n+1}) \leq & [(a_1 + sa_7 + a_{13})d(gz_{n-1}, gz_n) + (a_2 + sa_8 + a_{14})d(gx_{n-1}, gx_n) \\
& + (a_3 + sa_9 + a_{15})d(gy_{n-1}, gy_n)] \\
& + [(a_4 + sa_7)d(gz_n, gz_{n+1}) + (a_5 + sa_8)d(gx_n, gx_{n+1}) \\
& + (a_6 + sa_9)d(gy_n, gy_{n+1})].
\end{aligned} \quad (2.5)$$

令  $d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})$ . (2.3), (2.4) 与 (2.5) 式相加, 可得

$$d_n \leq (a_1 + a_2 + a_3 + sa_7 + sa_8 + sa_9 + a_{13} + a_{14} + a_{15})d_{n-1} + (a_4 + a_5 + a_6 + sa_7 + sa_8 + sa_9)d_n. \quad (2.6)$$

在 (2.1) 式, 中令  $x = x_n, y = y_n, z = z_n, u = x_{n-1}, v = y_{n-1}, w = z_{n-1}$ , 有

$$\begin{aligned}
d(gx_{n+1}, gx_n) = & d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
\leq & [a_1d(gx_n, F(x_n, y_n, z_n)) + a_2d(gy_n, F(y_n, z_n, x_n)) + a_3d(gz_n, F(z_n, x_n, y_n))] \\
& + [a_4d(gx_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})) + a_5d(gy_{n-1}, F(y_{n-1}, z_{n-1}, x_{n-1})) \\
& + a_6d(gz_{n-1}, F(z_{n-1}, x_{n-1}, y_{n-1}))] \\
& + [a_7d(gx_n, F(x_{n-1}, y_{n-1}, z_{n-1})) + a_8d(gy_n, F(y_{n-1}, z_{n-1}, x_{n-1})) \\
& + a_9d(gz_n, F(z_{n-1}, x_{n-1}, y_{n-1}))] \\
& + [a_{10}d(gx_{n-1}, F(x_n, y_n, z_n)) + a_{11}d(gy_{n-1}, F(y_n, z_n, x_n)) \\
& + a_{12}d(gz_{n-1}, F(z_n, x_n, y_n))] \\
& + [a_{13}d(gx_n, gx_{n-1}) + a_{14}d(gy_n, gy_{n-1}) + a_{15}d(gz_n, gz_{n-1})],
\end{aligned}$$

由 (2.2) 式可得

$$\begin{aligned}
d(gx_{n+1}, gx_n) = & d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
\leq & [a_1d(gx_n, gx_{n+1}) + a_2d(gy_n, gy_{n+1}) + a_3d(gz_n, gz_{n+1})] \\
& + [a_4d(gx_{n-1}, gx_n) + a_5d(gy_{n-1}, gy_n) + a_6d(gz_{n-1}, gz_n)] \\
& + [a_7d(gx_n, gx_n) + a_8d(gy_n, gy_n) + a_9d(gz_n, gz_n)] \\
& + [a_{10}d(gx_{n-1}, gx_{n+1}) + a_{11}d(gy_{n-1}, gy_{n+1}) + a_{12}d(gz_{n-1}, gz_{n+1})] \\
& + [a_{13}d(gx_n, gx_{n-1}) + a_{14}d(gy_n, gy_{n-1}) + a_{15}d(gz_n, gz_{n-1})] \\
\leq & [a_1d(gx_n, gx_{n+1}) + a_2d(gy_n, gy_{n+1}) + a_3d(gz_n, gz_{n+1})] \\
& + [a_4d(gx_{n-1}, gx_n) + a_5d(gy_{n-1}, gy_n) + a_6d(gz_{n-1}, gz_n)] \\
& + [sa_{10}(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})) + sa_{11}(d(gy_{n-1}, gy_n)d(gy_n, gy_{n+1})) \\
& + sa_{12}(d(gz_{n-1}, gz_n) + d(gz_n, gz_{n+1}))] \\
& + [a_{13}d(gx_n, gx_{n-1}) + a_{14}d(gy_n, gy_{n-1}) + a_{15}d(gz_n, gz_{n-1})],
\end{aligned}$$

从而有

$$\begin{aligned}
d(gx_{n+1}, gx_n) \leq & [(a_4 + sa_{10} + a_{13})d(gx_{n-1}, gx_n) + (a_5 + sa_{11} + a_{14})d(gy_{n-1}, gy_n) \\
& + (a_6 + sa_{12} + a_{15})d(gz_{n-1}, gz_n)] \\
& + [(a_1 + sa_{10})d(gx_n, gx_{n+1}) + (a_2 + sa_{11})d(gy_n, gy_{n+1}) \\
& + (a_3 + sa_{12})d(gz_n, gz_{n+1})].
\end{aligned} \quad (2.7)$$

同理可得

$$\begin{aligned} d(gy_{n+1}, gy_n) \preceq & [(a_4 + sa_{10} + a_{13})d(gy_{n-1}, gy_n) + (a_5 + sa_{11} + a_{14})d(gz_{n-1}, gz_n) \\ & + (a_6 + sa_{12} + a_{15})d(gx_{n-1}, gx_n)] \\ & + [(a_1 + sa_{10})d(gy_n, gy_{n+1}) + (a_2 + sa_{11})d(gz_n, gz_{n+1}) \\ & + (a_3 + sa_{12})d(gx_n, gx_{n+1})], \end{aligned} \quad (2.8)$$

$$\begin{aligned} d(gz_{n+1}, gz_n) \preceq & [(a_4 + sa_{10} + a_{13})d(gz_{n-1}, gz_n) + (a_5 + sa_{11} + a_{14})d(gx_{n-1}, gx_n) \\ & + (a_6 + sa_{12} + a_{15})d(gy_{n-1}, gy_n)] \\ & + [(a_1 + sa_{10})d(gz_n, gz_{n+1}) + (a_2 + sa_{11})d(gx_n, gx_{n+1}) \\ & + (a_3 + sa_{12})d(gy_n, gy_{n+1})]. \end{aligned} \quad (2.9)$$

(2.7), (2.8) 与 (2.9) 式相加, 可得

$$\begin{aligned} d_n \leq & (a_4 + a_5 + a_6 + sa_{10} + sa_{11} + sa_{12} + a_{13} + a_{14} + a_{15})d_{n-1} \\ & + (a_1 + a_2 + a_3 + sa_{10} + sa_{11} + sa_{12})d_n. \end{aligned} \quad (2.10)$$

由 (2.6) 与 (2.10) 式, 有

$$\begin{aligned} 2d_n \preceq & (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + sa_7 + sa_8 + sa_9 + sa_{10} + sa_{11} + sa_{12} + 2(a_{13} + a_{14} + a_{15}))d_{n-1} \\ & + (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + sa_7 + sa_8 + sa_9 + sa_{10} + sa_{11} + sa_{12})d_n, \end{aligned}$$

即

$$d_n \preceq hd_{n-1},$$

其中

$$h = \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + sa_7 + sa_8 + sa_9 + sa_{10} + sa_{11} + sa_{12} + 2(a_{13} + a_{14} + a_{15})}{2 - (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + sa_7 + sa_8 + sa_9 + sa_{10} + sa_{11} + sa_{12})} < \frac{1}{s}.$$

从而可得

$$d_n \preceq hd_{n-1} \preceq h^2d_{n-2} \preceq \cdots \preceq h^nd_0. \quad (2.11)$$

令  $m > n \geq 1$ , 有

$$d(gx_n, gx_m) \preceq sd(gx_n, gx_{n+1}) + s^2d(gx_{n+1}, gx_{n+2}) + \cdots + s^{m-n}d(gx_{m-1}, gx_m), \quad (2.12)$$

$$d(gy_n, gy_m) \preceq sd(gy_n, gy_{n+1}) + s^2d(gy_{n+1}, gy_{n+2}) + \cdots + s^{m-n}d(gy_{m-1}, gy_m), \quad (2.13)$$

$$d(gz_n, gz_m) \preceq sd(gz_n, gz_{n+1}) + s^2d(gz_{n+1}, gz_{n+2}) + \cdots + s^{m-n}d(gz_{m-1}, gz_m). \quad (2.14)$$

由 (2.11) 式和  $sh < 1$ , 有

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) & \preceq sd_n + s^2d_{n+1} + \cdots + s^{m-n}d_{m-1} \\ & \preceq sh^nd_0 + s^2h^{n+1}d_0 + \cdots + s^{m-n}h^{m-1}d_0 \\ & = (sh^n + s^2h^{n+1} + \cdots + s^{m-n}h^{m-1})d_0 \\ & = sh^n(1 + sh + (sh)^2 + \cdots + (sh)^{m-n-1})d_0 \\ & \preceq \frac{sh^n}{1 - sh}d_0 \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (2.15)$$

根据引理 1.4 (2) 可知, 对任意  $c \in E$ ,  $c \gg \theta$ , 存在  $N_0 \in N$ , 使得  $n > N_0$ ,  $\frac{sh^n}{1-sh}d_0 \ll c$ . 由 (2.15) 式和引理 1.4 (3) 可知, 对任意的  $m > n > N_0$ , 有  $d(gx_n, gx_m) + d(gx_n, gx_m) + d(gz_n, gz_m) \ll c$ , 从而可得

$$d(gx_n, gx_m) \ll c, \quad d(gx_n, gx_m) \ll c, \quad d(gz_n, gz_m) \ll c.$$

因此, 由定义 1.3 (2) 知  $\{gx_n\}, \{gy_n\}$  和  $\{gz_n\}$  是  $g(X)$  中的 Cauchy 列. 因为  $g(X)$  是完备的, 故存在  $x^*, y^*, z^* \in X$ , 使得  $gx_n \rightarrow gx^*$ ,  $gy_n \rightarrow gy^*$  和  $gz_n \rightarrow gz^*$ ,  $n \rightarrow \infty$ . 由定义 1.1 (3) 和 1.1 (1), 有

$$\begin{aligned} & d(F(x^*, y^*, z^*), gx^*) \\ & \leq s(d(F(x^*, y^*, z^*), gx_{n+1}) + d(gx_{n+1}, gx^*)) \\ & = s[d(F(x^*, y^*, z^*), F(x_n, y_n, z_n)) + d(gx_{n+1}, gx^*)] \\ & \leq s[a_1d(gx^*, F(x^*, y^*, z^*)) + a_2d(gy^*, F(y^*, z^*, x^*)) + a_3d(gz^*, F(z^*, x^*, y^*))] \\ & \quad + s[a_4d(gx_n, F(x_n, y_n, z_n)) + a_5d(gy_n, F(y_n, z_n, x_n)) + a_6d(gz_n, F(z_n, x_n, y_n))] \\ & \quad + s[a_7d(gx^*, F(x_n, y_n, z_n)) + a_8d(gy^*, F(y_n, z_n, x_n)) + a_9d(gz^*, F(z_n, x_n, y_n))] \\ & \quad + s[a_{10}d(gx_n, F(x^*, y^*, z^*)) + a_{11}d(gy_n, F(y^*, z^*, x^*)) + a_{12}d(gz_n, F(z^*, x^*, y^*))] \\ & \quad + s[a_{13}d(gx^*, gx_n) + a_{14}d(gy^*, gy_n) + a_{15}d(gz^*, gz_n)] + sd(gx_{n+1}, gx^*) \\ & \leq s[a_1d(gx^*, F(x^*, y^*, z^*)) + a_2d(gy^*, F(y^*, z^*, x^*)) + a_3d(gz^*, F(z^*, x^*, y^*))] \\ & \quad + s[sa_4d(gx_n, gx^*) + sa_4d(gx^*, gx_{n+1}) + sa_5d(gy_n, gy^*) + sa_5d(gy^*, gy_{n+1}) \\ & \quad + sa_6d(gz_n, gz^*) + sa_6d(gz^*, gz_{n+1})] \\ & \quad + s[a_7d(gx^*, gx_{n+1}) + a_8d(gy^*, gy_{n+1}) + a_9d(gz^*, gz_{n+1})] \\ & \quad + s[sa_{10}d(gx_n, gx^*) + sa_{10}d(gx^*, F(x^*, y^*, z^*)) + sa_{11}d(gy_n, gy^*) \\ & \quad + sa_{11}d(gy^*, F(y^*, z^*, x^*)) + sa_{12}d(gz_n, gz^*) + sa_{12}d(gz^*, F(z^*, x^*, y^*))] \\ & \quad + s[a_{13}d(gx^*, gx_n) + a_{14}d(gy^*, gy_n) + a_{15}d(gz^*, gz_n)] + sd(gx_{n+1}, gx^*). \end{aligned}$$

从而可得

$$\begin{aligned} d(F(x^*, y^*, z^*), gx^*) & \leq (sa_1 + s^2a_{10})d(F(x^*, y^*, z^*), gx^*) + (sa_2 + s^2a_{11})d(F(y^*, z^*, x^*), gy^*) \\ & \quad + (sa_3 + s^2a_{12})d(F(z^*, x^*, y^*), gz^*) + (s^2a_4 + s^2a_{10} + sa_{13})d(gx_n, gx^*) \\ & \quad + (s^2a_4 + s^2a_5 + s)d(gx_{n+1}, gx^*) + (s^2a_6 + s^2a_{12} + sa_{15})d(gz_n, gz^*) \\ & \quad + (s^2a_5 + s^2a_{11} + sa_{14})d(gy_n, gy^*) + (s^2a_5 + sa_8)d(gy_{n+1}, gy^*) \\ & \quad + (s^2a_6 + sa_9)d(gz_{n+1}, gz^*). \end{aligned} \quad (2.16)$$

同理可得

$$\begin{aligned} d(F(y^*, z^*, x^*), gy^*) & \leq (sa_1 + s^2a_{10})d(F(y^*, z^*, x^*), gy^*) + (sa_2 + s^2a_{11})d(F(z^*, x^*, y^*), gz^*) \\ & \quad + (sa_3 + s^2a_{12})d(F(x^*, y^*, z^*), gx^*) + (s^2a_4 + s^2a_{10} + sa_{13})d(gy_n, gy^*) \\ & \quad + (s^2a_4 + s^2a_5 + s)d(gy_{n+1}, gy^*) + (s^2a_6 + s^2a_{12} + sa_{15})d(gx_n, gx^*) \\ & \quad + (s^2a_5 + s^2a_{11} + sa_{14})d(gz_n, gz^*) + (s^2a_5 + sa_8)d(gz_{n+1}, gz^*) \\ & \quad + (s^2a_6 + sa_9)d(gx_{n+1}, gx^*). \end{aligned} \quad (2.17)$$

$$\begin{aligned}
d(F(z^*, x^*, y^*), gz^*) &\preceq (sa_1 + s^2a_{10})d(F(z^*, x^*, y^*), gz^*) + (sa_2 + s^2a_{11})d(F(x^*, y^*, z^*), gx^*) \\
&\quad + (sa_3 + s^2a_{12})d(F(y^*, z^*, x^*), gy^*) + (s^2a_4 + s^2a_{10} + sa_{13})d(gz_n, gz^*) \\
&\quad + (s^2a_4 + s^2a_5 + s)d(gz_{n+1}, gz^*) \\
&\quad + (s^2a_6 + s^2a_{12} + sa_{15})d(gy_n, gy^*) + (s^2a_5 + s^2a_{11} + sa_{14})d(gx_n, gx^*) \\
&\quad + (s^2a_5 + sa_8)d(gx_{n+1}, gx^*) + (s^2a_6 + sa_9)d(gy_{n+1}, gy^*). \quad (2.18)
\end{aligned}$$

令  $\delta = d(F(x^*, y^*, z^*), gx^*) + d(F(y^*, z^*, x^*), gy^*) + d(F(z^*, x^*, y^*), gz^*)$ . (2.16), (2.17) 与 (2.18) 式相加, 可得

$$\begin{aligned}
\delta &\preceq (sa_1 + s^2a_{10} + sa_2 + s^2a_{11} + sa_3 + s^2a_{12})\delta \\
&\quad + (s^2a_4 + s^2a_{10} + sa_{13} + s^2a_6 + s^2a_{12} + sa_{15} + s^2a_5 + s^2a_{11} + sa_{14})d(gx_n, gx^*) \\
&\quad + (s^2a_4 + s^2a_{10} + sa_{13} + s^2a_5 + s^2a_{11} + sa_{14} + s^2a_6 + s^2a_{12} + sa_{15})d(gy_n, gy^*) \\
&\quad + (s^2a_4 + sa_7 + s + s^2a_6 + sa_9 + s^2a_5 + sa_8)d(gx_{n+1}, gx^*) \\
&\quad + (s^2a_4 + sa_7 + s + s^2a_5 + sa_8 + s^2a_6 + sa_9)d(gy_{n+1}, gy^*) \\
&\quad + (s^2a_6 + s^2a_{12} + sa_{15} + s^2a_5 + s^2a_{11} + sa_{14} + s^2a_4 + s^2a_{10} + sa_{13})d(gz_n, gz^*) \\
&\quad + (s^2a_4 + sa_7 + s + s^2a_5 + sa_8 + s^2a_6 + sa_9)d(gz_{n+1}, gz^*).
\end{aligned}$$

那么有

$$\begin{aligned}
\delta &\preceq \frac{B}{1-A}d(gx_n, gx^*) + \frac{B}{1-A}d(gy_n, gy^*) + \frac{B}{1-A}d(gz_n, gz^*) \\
&\quad + \frac{C}{1-A}d(gx_{n+1}, gx^*) + \frac{C}{1-A}d(gy_{n+1}, gy^*) + \frac{C}{1-A}d(gz_{n+1}, gz^*),
\end{aligned}$$

其中  $A = sa_1 + s^2a_{10} + sa_2 + s^2a_{11} + sa_3 + s^2a_{12}$ ,  $B = s^2a_4 + s^2a_{10} + sa_{13} + s^2a_6 + s^2a_{12} + sa_{15} + s^2a_5 + s^2a_{11} + sa_{14}$ ,  $C = s^2a_4 + sa_7 + s + s^2a_6 + sa_9 + s^2a_5 + sa_8$ . 因为  $gx_n \rightarrow gx^*$ ,  $gy_n \rightarrow gy^*$ ,  $gz_n \rightarrow gz^*$ ,  $n \rightarrow \infty$ , 那么, 由定义 1.3 (1) 及  $c \gg \theta$  可知, 存在  $N_0 \in \mathbb{N}$ , 使得对任意的  $n > N_0$ ,

$$\begin{aligned}
d(gx_n, gx^*) &\ll c \frac{1-A}{6B}, \quad d(gy_n, gy^*) \ll c \frac{1-A}{6B}, \quad d(gz_n, gz^*) \ll c \frac{1-A}{6B}, \\
d(gx_{n+1}, gx^*) &\ll c \frac{1-A}{6C}, \quad d(gy_{n+1}, gy^*) \ll c \frac{1-A}{6C}, \quad d(gz_{n+1}, gz^*) \ll c \frac{1-A}{6C},
\end{aligned}$$

从而有

$$\begin{aligned}
\delta &\preceq \frac{B}{1-A}d(gx_n, gx^*) + \frac{B}{1-A}d(gy_n, gy^*) + \frac{B}{1-A}d(gz_n, gz^*) \\
&\quad + \frac{C}{1-A}d(gx_{n+1}, gx^*) + \frac{C}{1-A}d(gy_{n+1}, gy^*) + \frac{C}{1-A}d(gz_{n+1}, gz^*) \\
&\ll c \frac{1-A}{6B} \frac{B}{1-A} + c \frac{1-A}{6B} \frac{B}{1-A} + c \frac{1-A}{6B} \frac{B}{1-A} \\
&\quad + c \frac{1-A}{6C} \frac{C}{1-A} + c \frac{1-A}{6C} \frac{C}{1-A} + c \frac{1-A}{6C} \frac{C}{1-A} \\
&= \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} = c.
\end{aligned}$$

证毕.



根据引理 1.4 (4) 有  $\delta = \theta$ , 即

$$d(F(x^*, y^*, z^*), gx^*) + d(F(y^*, z^*, x^*), gy^*) + d(F(z^*, x^*, y^*), gz^*) = \theta.$$

故可得  $d(F(x^*, y^*, z^*), gx^*) = \theta$ ,  $d(F(y^*, z^*, x^*), gy^*) = \theta$ ,  $d(F(z^*, x^*, y^*), gz^*) = \theta$ . 因此  $gx^* = F(x^*, y^*, z^*)$ ,  $gy^* = F(y^*, z^*, x^*)$ ,  $gz^* = F(z^*, x^*, y^*)$ . 故  $(x^*, y^*, z^*)$  是  $F$  和  $g$  的三元重合点.

**注 2.6** 定理 2.5 推广了文 [6, 定理 2.1].

**推论 2.7** 设  $(X, d)$  是锥 b- 度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F: X \times X \times X \rightarrow X$ ,  $g: X \rightarrow X$ , 假设存在非负常数  $k, l, t$  且  $k + l + t < \frac{1}{s}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(gx, gu) + ld(gy, gv) + td(gz, gw).$$

若  $F(X^3) \subseteq g(X)$ , 并且  $g(X)$  是  $X$  中完备子空间, 则  $F$  和  $g$  有三元重合点  $(x^*, y^*, z^*) \in X^3$ .

**推论 2.8** 设  $(X, d)$  是锥 b- 度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F: X \times X \times X \rightarrow X$ ,  $g: X \rightarrow X$ , 假设存在非负常数  $k, l$  且  $k + l < \frac{2}{s(s+1)}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(gx, F(u, v, w)) + ld(gu, F(x, y, z)).$$

若  $F(X^3) \subseteq g(X)$ , 并且  $g(X)$  是  $X$  中完备子空间, 则  $F$  和  $g$  有三元重合点  $(x^*, y^*, z^*) \in X^3$ .

**推论 2.9** 设  $(X, d)$  是锥 b- 度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F: X \times X \times X \rightarrow X$ ,  $g: X \rightarrow X$ , 假设存在非负常数  $k, l$ , 且  $k + l < \frac{2}{s+1}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(gx, F(x, y, z)) + ld(gu, F(u, v, w)).$$

若  $F(X^3) \subseteq g(X)$ , 并且  $g(X)$  是  $X$  中完备子空间, 则  $F$  和  $g$  有三元重合点  $(x^*, y^*, z^*) \in X^3$ .

### 3 锥 b- 度量空间中的三元不动点

**定理 3.1** 在定理 2.5 中, 如果  $F$  和  $g$  还满足弱相容条件, 则  $F$  和  $g$  有唯一的三元公共不动点.

**证明** 由定理 2.5 可知  $F$  和  $g$  有三元重合点  $(x^*, y^*, z^*)$ , 即

$$gx^* = F(x^*, y^*, z^*), \quad gy^* = F(y^*, z^*, x^*), \quad gz^* = F(z^*, x^*, y^*).$$

下证唯一性. 假设  $F$  和  $g$  有另外一个三元重合点  $(gx', gy', gz')$ , 使得  $gx' = F(x', y', z')$ ,  $gy' = F(y', z', x')$ ,  $gz' = F(z', x', y')$ ,  $(x', y', z') \in X^3$ . 由 (2.1) 式有

$$\begin{aligned} d(gx^*, gx') &= d(F(x^*, y^*, z^*), F(x', y', z')) \\ &\preceq [a_1 d(gx^*, F(x^*, y^*, z^*)) + a_2 d(gy^*, F(y^*, z^*, x^*)) + a_3 d(gz^*, F(z^*, x^*, y^*))] \\ &\quad + [a_4 d(gx', F(x', y', z')) + a_5 d(gy', F(y', z', x')) + a_6 d(gz', F(z', x', y'))] \\ &\quad + [a_7 d(gx^*, F(x', y', z')) + a_8 d(gy^*, F(y', z', x')) + a_9 d(gz^*, F(z', x', y'))] \\ &\quad + [a_{10} d(gx', F(x^*, y^*, z^*)) + a_{11} d(gy', F(y^*, z^*, x^*)) + a_{12} d(gz', F(z^*, x^*, y^*))] \\ &\quad + [a_{13} d(gx^*, gx') + a_{14} d(gy^*, gy') + a_{15} d(gz^*, gz')] \\ &= [a_1 d(gx^*, gx^*) + a_2 d(gy^*, gy^*) + a_3 d(gz^*, gz^*)] \\ &\quad + [a_4 d(gx', gx') + a_5 d(gy', gy') + a_6 d(gz', gz')] \\ &\quad + [a_7 d(gx^*, gx') + a_8 d(gy^*, gy') + a_9 d(gz^*, gz')] \\ &\quad + [a_{10} d(gx', gx^*) + a_{11} d(gy', gy^*) + a_{12} d(gz', gz^*)] \\ &\quad + [a_{13} d(gx^*, gx') + a_{14} d(gy^*, gy') + a_{15} d(gz^*, gz')] \end{aligned}$$

$$\begin{aligned}
&= [a_7 d(gx^*, gx') + a_8 d(gy^*, gy') + a_9 d(gz^*, gz')] \\
&\quad + [a_{10} d(gx', gx^*) + a_{11} d(gy', gy^*) + a_{12} d(gz', gz^*)] \\
&\quad + [a_{13} d(gx^*, gx') + a_{14} d(gy^*, gy') + a_{15} d(gz^*, gz')].
\end{aligned}$$

从而可得

$$\begin{aligned}
d(gx^*, gx') &\leq (a_7 + a_{10} + a_{13})d(gx^*, gx') + (a_8 + a_{11} + a_{14})d(gy^*, gy') \\
&\quad + (a_9 + a_{12} + a_{15})d(gz^*, gz'),
\end{aligned} \tag{3.1}$$

同理可得

$$\begin{aligned}
d(gy^*, gy') &\leq (a_7 + a_{10} + a_{13})d(gy^*, gy') + (a_8 + a_{11} + a_{14})d(gz^*, gz') \\
&\quad + (a_9 + a_{12} + a_{15})d(gx^*, gx'),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
d(gz^*, gz') &\leq (a_7 + a_{10} + a_{13})d(gz^*, gz') + (a_8 + a_{11} + a_{14})d(gx^*, gx') \\
&\quad + (a_9 + a_{12} + a_{15})d(gy^*, gy').
\end{aligned} \tag{3.3}$$

由 (3.1), (3.2) 和 (3.3) 式相加, 可得

$$\begin{aligned}
&d(gx^*, gx') + d(gy^*, gy') + d(gz^*, gz') \\
&\leq (a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15})(d(gx^*, gx') + d(gy^*, gy') + d(gz^*, gz')).
\end{aligned}$$

因为  $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} < 1$ , 故由引理 1.4 (1), 可得

$$d(gx^*, gx') + d(gy^*, gy') + d(gz^*, gz') = \theta.$$

又因  $d(gx^*, gx') \geq \theta$ ,  $d(gy^*, gy') \geq \theta$ ,  $d(gz^*, gz') \geq \theta$ . 因此有

$$d(gx^*, gx') = \theta, \quad d(gy^*, gy') = \theta, \quad d(gz^*, gz') = \theta,$$

即

$$gx^* = gx', \quad gy^* = gy', \quad gz^* = gz'. \tag{3.4}$$

唯一性得证. 同理可得

$$gx^* = gy', \quad gy^* = gz', \quad gz^* = gx'. \tag{3.5}$$

$$gx^* = gz', \quad gy^* = gx', \quad gz^* = gy'. \tag{3.6}$$

由 (3.4), (3.5) 和 (3.6) 式, 可得

$$gx^* = gy^* = gz^*,$$

即  $(gx^*, gx^*, gx^*)$  是  $F$  和  $g$  的唯一三元重合点. 令  $u = gx^* = F(x^*, y^*, z^*)$ . 由  $F$  和  $g$  的弱相容性, 有

$$gu = g(gx^*) = gF(x^*, y^*, z^*) = F(gx^*, gy^*, gz^*) = F(gx^*, gx^*, gx^*) = F(u, u, u).$$

那么  $(gu, gu, gu)$  是  $F$  和  $g$  的三元重合点, 又因为  $(u, u, u)$  也是  $F$  和  $g$  的三元重合点, 故由重合点的唯一性可知  $gu = u$ . 从而可得  $u = gu = F(u, u, u)$ . 因此,  $(u, u, u)$  是  $F$  和  $g$  的唯一三元公共不动点. 证毕.

下面, 通过一个例子来给出定理 2.5 和 3.1 的应用.

**例 3.2** 令  $X = [0, 1]$ ,  $E = C_R^1[0, 1]$ ,  $\|u\| = \|u\|_\infty + \|u'\|_\infty$ ,  $u \in E$ ,  $P = \{u \in E : u(t) \geq 0\}$ , 易知  $P$  是体锥且不是正规的. 定义锥 b- 度量  $d: X \times X \rightarrow E$  为  $d(x, y)(t) = |x - y|^2 e^t$ , 则  $(X, d)$  为完备的锥 b- 度量空间, 并且  $s = 2$ . 定义  $F: X \times X \times X \rightarrow X$ ,  $g: X \rightarrow X$  为

$$F(x, y, z) = \frac{1}{11}x + \frac{1}{10}y + \frac{1}{9}z, \quad g(x) = \frac{1}{2}x, \quad \forall x \in X.$$

从而有

$$\begin{aligned} d(F(x, y, z), F(u, v, w))(t) &= \left| \frac{1}{11}x + \frac{1}{10}y + \frac{1}{9}z - \frac{1}{11}u - \frac{1}{10}v - \frac{1}{9}w \right|^2 e^t \\ &= \left| \left( \frac{1}{11}x - \frac{1}{11}u \right) + \left( \frac{1}{10}y - \frac{1}{10}v \right) + \left( \frac{1}{9}z - \frac{1}{9}w \right) \right|^2 e^t \\ &= \left| \frac{2}{11} \left( \frac{1}{2}x - \frac{1}{2}u \right) + \frac{2}{10} \left( \frac{1}{2}y - \frac{1}{2}v \right) + \frac{2}{9} \left( \frac{1}{2}z - \frac{1}{2}w \right) \right|^2 e^t \\ &\leq 4 \left( \left| \frac{2}{11} \left( \frac{1}{2}x - \frac{1}{2}u \right) \right|^2 e^t + \left| \frac{2}{10} \left( \frac{1}{2}y - \frac{1}{2}v \right) \right|^2 e^t + \left| \frac{2}{9} \left( \frac{1}{2}z - \frac{1}{2}w \right) \right|^2 e^t \right) \\ &= \frac{16}{121} \left| \frac{1}{2}x - \frac{1}{2}u \right|^2 e^t + \frac{16}{100} \left| \frac{1}{2}y - \frac{1}{2}v \right|^2 e^t + \frac{16}{81} \left| \frac{1}{2}z - \frac{1}{2}w \right|^2 e^t \\ &= \frac{16}{121} d(gx, gu)(t) + \frac{16}{100} d(gy, gv)(t) + \frac{16}{81} d(gz, gw)(t), \end{aligned}$$

其中

$$a_{13} = \frac{16}{121}, \quad a_{14} = \frac{16}{100}, \quad a_{15} = \frac{16}{81}, \quad a_i = 0, \quad i = 1, 2, \dots, 12.$$

注意到

$$2s(a_{13} + a_{14} + a_{15}) = 4 \left( \frac{16}{121} + \frac{16}{100} + \frac{16}{81} \right) < 2, \quad F(X \times X \times X) \subseteq g(X)$$

是  $X$  的完备子空间. 因此满足定理 2.5 的条件, 故  $F$  和  $g$  有一个三元重合点  $(0, 0, 0)$ . 又因为  $F$  和  $g$  在  $(0, 0, 0)$  是弱相容的, 故由定理 3.1 可得,  $(0, 0, 0)$  是  $F$  和  $g$  唯一的三元公共不动点.

在定理 3.1 中令  $g = I$  ( $I$  为恒等映射), 则下面结论成立.

**定理 3.3** 设  $(X, d)$  是锥 b- 度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F: X \times X \times X \rightarrow X$ , 假设存在非负常数  $a_i \in [0, 1]$ ,  $i = 1, 2, \dots, 15$ ,  $(s+1)(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + s(s+1)(a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12}) + 2s(a_{13} + a_{14} + a_{15}) < 2$ , 并且  $\sum_{i=1}^{15} a_i < 1$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) &\leq [a_1 d(x, F(x, y, z)) + a_2 d(y, F(y, z, x)) + a_3 d(z, F(z, x, y))] \\ &\quad + [a_4 d(u, F(u, v, w)) + a_5 d(v, F(v, w, u)) + a_6 d(w, F(w, u, v))] \\ &\quad + [a_7 d(x, F(u, v, w)) + a_8 d(y, F(v, w, u)) + a_9 d(z, F(w, u, v))] \\ &\quad + [a_{10} d(u, F(x, y, z)) + a_{11} d(v, F(y, z, x)) + a_{12} d(w, F(z, x, y))] \\ &\quad + [a_{13} d(x, u) + a_{14} d(y, v) + a_{15} d(z, w)], \end{aligned}$$

则  $F$  存在唯一的三元不动点  $(x^*, x^*, x^*) \in X^3$ .

**推论 3.4** 设  $(X, d)$  是锥  $b$ -度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F : X \times X \times X \rightarrow X$ , 假设存在非负常数  $k, l, t$  且  $k + l + t < \frac{1}{s}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(x, u) + ld(y, v) + td(z, w),$$

则  $F$  存在唯一的三元不动点  $(x^*, x^*, x^*) \in X^3$ .

**推论 3.5** 设  $(X, d)$  是锥  $b$ -度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F : X \times X \times X \rightarrow X$ , 假设存在非负常数  $k, l$  且  $k + l < \frac{2}{s(s+1)}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(x, F(u, v, w)) + ld(u, F(x, y, z)),$$

则  $F$  存在唯一的三元不动点  $(x^*, x^*, x^*) \in X^3$ .

**推论 3.6** 设  $(X, d)$  是锥  $b$ -度量空间,  $s \geq 1$ ,  $P$  是体锥. 设  $F : X \times X \times X \rightarrow X$ , 假设存在非负常数  $k, l$  且  $k + l < \frac{2}{s+1}$ , 使得对任意的  $x, y, z, u, v, w \in X$ , 有

$$d(F(x, y, z), F(u, v, w)) \preceq kd(x, F(x, y, z)) + ld(u, F(u, v, w)),$$

则  $F$  存在唯一的三元不动点  $(x^*, x^*, x^*) \in X^3$ .

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