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Fermat 型偏微差分方程组的 整函数解

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摘 要 利用多变量 Nevanlinna 值分布理论与 Nevanlinna 理论的差分模拟结果, 讨论了几类多变量复域 Fermat 型偏微差分方程组解的性质, 得到了方程组有限超越整函数解的存在性条件与具体形式, 推广改进了高凌云、曹廷彬、刘凯等人的结果, 给出例子说明多变量与单变量方程组有限级超越整函数解之间的差异.

关键词 整函数; 偏微差分方程组; 存在性

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Entire Solutions of Several Fermat Type Systems of Partial Differential Difference Equations

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Abstract By making use of the Nevanlinna theory and difference Nevanlinna theory of several complex variables, we investigate several Fermat type systems of partial differential difference equations, and obtain a series of results about the existence and the forms of entire solutions of such systems, which are some improvements and gen-

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eralization of the previous results given by Cao, Gao, Liu et al. We also give some examples to show that there exists significant differences in the forms of transcendental entire solutions with finite order of the systems of the equations with between several complex variables and single complex variable.

Keywords entire function; systems of partial differential difference equations; existence

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1 引言

1995 年, Wiles^[25, 26] 证明了著名的 Fermat 大定理: 当正整数 $m > 2$ 时, 方程 $x^m + y^m = z^m$ 没有正整数解. 该定理又称 Fermat 最后的定理, 由 17 世纪法国数学家 Pierre de Fermat 提出. 他们还指出, 当 $m \geq 3$ 时, 方程 $x^m + y^m = 1$ 没有非平凡的有理数解; 当 $m = 2$ 时, 该方程存在非平凡的有理数解.

对于 Fermat 型函数方程 $f(z)^m + g(z)^m = 1$, Montel^[20] 指出: 当 $m \geq 3$ 时, 此方程没有非常数整函数解; Gross^[7] 证明: 当 $m = 2$ 时, 此方程具有整函数形如 $f(z) = \cos(p(z))$, $g(z) = \sin(p(z))$ 的解, 这里 $p(z)$ 为整函数. 对于更一般形式的函数方程 $f(z)^n + g(z)^m = 1$, Yang^[28] 证明: 若正整数 n, m 满足 $\frac{1}{n} + \frac{1}{m} < 1$, 方程不存在非常数整函数解 f, g . 若 $g(z)$ 为 $f(z)$ 的某种特殊形式, Yang 和 Li^[29] 考虑了方程

$$f(z)^2 + f'(z)^2 = 1, \quad (1.1)$$

并得到方程 (1.1) 的超越亚纯解具有形式 $f(z) = \frac{1}{2}(Pe^{\lambda z} + \frac{1}{P}e^{-\lambda z})$, 其中 P, λ 为非零常数.

近年来, 随着 Nevanlinna 理论差分模拟结果的建立, 复域差分、微差分方程引起了很多研究者的关注, 在复差分算子、微(差)分方程解的相关性质方面得到了许多重要且有趣的结果(见文 [5, 8–10, 22]). Liu, Yang^[15–18] 等考虑了几类涉及微分、差分的 Fermat 型函数方程

$$f(z)^2 + f(z+c)^2 = 1, \quad (1.2)$$

$$f'(z)^2 + f(z+c)^2 = 1, \quad (1.3)$$

$$f'(z)^2 + [f(z+c) - f(z)]^2 = 1, \quad (1.4)$$

这里 $c \in \mathbb{C}$ 为非零复常数, 并得到方程 (1.2) 的超越整函数解 $f(z)$ 具有形式 $f(z) = \sin(Az + B)$, 其中 $A = \frac{(4k+1)\pi}{2c}$, 这里 $B \in \mathbb{C}$, $k \in \mathbb{N}$; 方程 (1.3) 的超越整函数解 $f(z)$ 具有形式 $f(z) = \sin(z \pm Bi)$, 其中 $c = k\pi$; 方程 (1.3) 的超越整函数解 $f(z)$ 具有形式 $f(z) = \frac{1}{2}\sin(2z + Bi)$, 其中 $c = k\pi + \frac{\pi}{2}$.

最近, 高凌云^[6, 19] 等将方程 (1.3)–(1.4) 推广至方程组情形, 得到

定理 A^[6] 若 (f_1, f_2) 为微差分方程组

$$\begin{cases} [f_1'(z)]^2 + f_2(z+c)^2 = 1, \\ [f_2'(z)]^2 + f_1(z+c)^2 = 1 \end{cases}$$

的有限级超越整函数解, 则 (f_1, f_2) 具有形式

$$(f_1(z), f_2(z)) = (\sin(z - bi), \sin(z - b_1i)) \text{ 或 } (f_1(z), f_2(z)) = (\sin(z + bi), \sin(z + b_1i)),$$

其中 $b, b_1 \in \mathbb{C}$ 为复常数, $c = k\pi$.

定理 B ^[19] 设 (f_1, f_2) 是复微分 - 差分方程组

$$\begin{cases} f_1'(z)^2 + [f_2(z+c) - f_1(z)]^2 = Q_1(z), \\ f_2'(z)^2 + [f_1(z+c) - f_2(z)]^2 = Q_2(z) \end{cases}$$

有限级超越整函数解, 则有 $Q_1(z) = c_{11}c_{12}$ 和 $Q_2(z) = c_{21}c_{22}$ 均为常数, 且 (f_1, f_2) 满足

$$(f_1, f_2) = \left(\frac{c_{11}e^{az+b_1} - c_{12}e^{-az-b_1}}{2a}, \frac{c_{21}e^{az+b_2} - c_{22}e^{-az-b_2}}{2a} \right),$$

其中 $(a+i)^4 = 1$, b_1 和 b_2 为常数, $c = -\frac{\log(-(a+i)^2)+2k\pi i}{2a}$, $k \in \mathbb{N}$.

Xu 与 Cao ^[2, 3, 27] 利用多变量 Nevanlinna 差分理论 ^[4], 讨论了多变量 Fermat 型偏微差分方程解的形式与存在性, 得到

定理 C ^[27] 设 $c = (c_1, c_2) \in \mathbb{C}^2$, 则 Fermat 型偏微差分方程

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1} \right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

的有限级超越整函数解 $f(z_1, z_2)$ 具有形式 $f(z_1, z_2) = \sin(Az_1 + Bz_2 + H(z_2))$, 其中 A, B 为 \mathbb{C} 中复常数, $A^2 = 1$, $Ae^{i(Ac_1+Bc_2)} = 1$, $H(z_2)$ 为仅关于 z_2 的多项式且 $H(z_2) = H(z_2 + c_2)$.

定理 D ^[2] 设 $c = (c_1, c_2) \in \mathbb{C}^2$, 若 $n \neq m$, $n, m \in \mathbb{N}_+$, 则 Fermat 型偏微差分方程

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1} \right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1$$

不存在任何有限级超越整函数解.

综合上述结果, 我们提出下列问题:

问题 1.1 若定理 C 与 D 中方程同时包含 $\frac{\partial f(z_1, z_2)}{\partial z_1}$ 与 $\frac{\partial f(z_1, z_2)}{\partial z_2}$, 方程解的存在性与形式情况如何?

问题 1.2 若定理 C 与 D 中方程转化为方程组, 其超越整函数解的存在性与形式又如何?

2 主要结果

本文从上述问题出发, 讨论了几类 Fermat 型偏微差分方程组解的存在性与其形式. 如无特别说明, 均记 $z = (z_1, z_2)$, $L(z) := a_1z_1 + a_2z_2$, $L(c) := a_1c_1 + a_2c_2$, a_1, a_2, c_1, c_2 为 \mathbb{C} 上复常数.

定理 2.1 设 $c = (c_1, c_2) \in \mathbb{C}^2$ 为复常数, m_j, n_j ($j = 1, 2$) 为正整数, α_j, β_j ($j = 1, 2$) $\in \mathbb{C}$, 且 α_1, α_2 不同时为零, β_1, β_2 也不同时为零. 那么, 复偏微差分方程组

$$\begin{cases} \left(\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} \right)^{m_1} + f_2(z_1 + c_1, z_2 + c_2)^{n_1} = 1, \\ \left(\beta_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \beta_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} \right)^{m_2} + f_1(z_1 + c_1, z_2 + c_2)^{n_2} = 1 \end{cases} \quad (2.1)$$

不存在有限级超越整函数解, 当下列条件之一成立:

- (i) $n_1n_2 > m_1m_2$;
- (ii) $m_j > \frac{n_j}{n_j-1}$ 且 $n_j \geq 2$, $j = 1, 2$.

注 2.1 若 f, g 为超越整函数, $\rho = \max\{\rho(f), \rho(g)\} < \infty$, 且满足方程组

$$\begin{cases} f^{m_1} + g^{n_1} = 1, \\ f^{m_2} + g^{n_2} = 1, \end{cases} \quad (2.2)$$

则称 (f, g) 是方程组 (2.2) 的有限级超越整函数解.

注 2.2 下列例子说明若 $n_1 n_2 < m_1 m_2$ 或 $m_j < \frac{n_j}{n_j - 1}$ 成立时, 方程组 (2.1) 存在有限级超越整函数解.

例 2.1 若 $m_1 = m_2 = 2, n_1 = n_2 = 1$, 方程组

$$\begin{cases} \left(\frac{\partial f_1(z_1, z_2)}{\partial z_1} + \frac{\partial f_1(z_1, z_2)}{\partial z_2} \right)^2 + f_2(z_1 + 1, z_2 + 2) = 1, \\ \left(\frac{\partial f_2(z_1, z_2)}{\partial z_1} + \frac{\partial f_2(z_1, z_2)}{\partial z_2} \right)^2 + f_1(z_1 + 1, z_2 + 2) = 1 \end{cases}$$

存在如下有限级超越整函数解 $(f_1(z_1, z_2), f_2(z_1, z_2))$,

$$\begin{cases} f_1 = \frac{5 - z_1^2}{4} + \frac{1}{2}(z_2 - z_1)(z_1 - 1) - \frac{1}{4}[(z_2 - z_1) - 1]^2 + e^{\pi i(z_2 - z_1)}(2z_1 - z_2) - e^{2\pi i(z_2 - z_1)}, \\ f_2 = \frac{5 - z_1^2}{4} + \frac{1}{2}(z_2 - z_1)(z_1 - 1) - \frac{1}{4}[(z_2 - z_1) - 1]^2 - e^{\pi i(z_2 - z_1)}(2z_1 - z_2) - e^{2\pi i(z_2 - z_1)}. \end{cases}$$

当 $m_1 = m_2 = 2, n_1 = n_2 = 2$ 时, 关于方程组 (2.1) 有限超越整函数解的存在性与形式, 有

定理 2.2 若 $c = (c_1, c_2)$ 是 \mathbb{C}^2 上复常数, α_j ($j = 1, 2, 3$) 为 \mathbb{C} 上复常数, 且 α_1, α_2 不同时为零, $\alpha_3 \neq 0$. 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组

$$\begin{cases} \left(\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} \right)^2 + [\alpha_3 f_2(z_1 + c_1, z_2 + c_2)]^2 = 1, \\ \left(\alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} \right)^2 + [\alpha_3 f_1(z_1 + c_1, z_2 + c_2)]^2 = 1 \end{cases} \quad (2.3)$$

的有限级超越整函数解, 则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有形式

$$(f_1, f_2) = \left(\frac{e^{L(z)+H(z)+B_0} - e^{-(L(z)+H(z)+B_0)}}{2(\alpha_1 a_1 + \alpha_2 a_2)}, \frac{A_{21} e^{L(z)+H(z)+B_0} + A_{22} e^{-(L(z)+H(z)+B_0)}}{2\alpha_3} \right),$$

其中 $H(z) := H(s_1)$ 是关于 $s_1 := c_2 z_1 - c_1 z_2$ 的多项式, $(\alpha_2 c_1 - \alpha_1 c_2)H' \equiv 0$, $(\alpha_1 a_1 + \alpha_2 a_2)^2 = -\alpha_3^2$, $e^{2L(c)} = \pm 1$, $B_0 \in \mathbb{C}$, 且 c_1, c_2, A_{21}, A_{22} 满足下列情形之一:

- (i) $e^{L(c)} = 1, A_{21} = -i, A_{22} = i$;
- (ii) $e^{L(c)} = -1, A_{21} = i, A_{22} = -i$;
- (iii) $e^{L(c)} = i, A_{21} = -1, A_{22} = -1$;
- (iv) $e^{L(c)} = -i, A_{21} = 1, A_{22} = 1$.

注 2.3 由定理 2.2 知, 当 $\alpha_2 c_1 - \alpha_1 c_2 \neq 0$ 时, 方程组仅存在 $\rho = \max\{\rho(f_1), \rho(f_2)\} = 1$ 的有限级超越整函数解; 当 $\alpha_2 c_1 - \alpha_1 c_2 = 0$ 时, 方程组存在级 $\rho = \max\{\rho(f_1), \rho(f_2)\} \geq 1$ 的有限级超越整函数解 (见例 2.3 与 2.4), 与单变量 Fermat 型方程组解的形式具有显著差异 (定理 A 和 B 只存在级为 1 的有限级超越整函数解).

下列例子说明方程组 (2.3) 在各种情形下解的存在性.

例 2.2 若

$$(f_1, f_2) = \left(\frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{-2i}, \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.3) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, c_1 = \pi, c_2 = -\pi$.

例 2.3 若

$$(f_1, f_2) = \left(\frac{e^{i(\frac{z_1}{2}+z_2)+\pi^3(z_2-\frac{z_1}{2})^3} - e^{-i(\frac{z_1}{2}+z_2)-\pi^3(z_2-\frac{z_1}{2})^3}}{4i}, \frac{-e^{i(\frac{z_1}{2}+z_2)+\pi^3(z_2-\frac{z_1}{2})^3} - e^{-i(\frac{z_1}{2}+z_2)-\pi^3(z_2-\frac{z_1}{2})^3}}{4i} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.3) 的一对有限级超越整函数解, 其中 $\alpha_1 = 2, \alpha_2 = 1, \alpha_3 = 2, c_1 = \pi, c_2 = \frac{1}{2}\pi$.

例 2.4 若

$$(f_1, f_2) = \left(\frac{e^{i(z_1-z_2)+\frac{\pi^2}{16}(z_1+z_2)^2} - e^{-i(z_1-z_2)-\frac{\pi^2}{16}(z_1+z_2)^2}}{4i}, \frac{e^{i(z_1-z_2)+\frac{\pi^2}{16}(z_1+z_2)^2} + e^{-i(z_1-z_2)-\frac{\pi^2}{16}(z_1+z_2)^2}}{4i} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.3) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1, c_1 = \frac{\pi}{4}, c_2 = -\frac{\pi}{4}$.

例 2.5 若

$$(f_1, f_2) = \left(\frac{e^{\frac{i}{2}(z_1+z_2)} - e^{-\frac{i}{2}(z_1+z_2)}}{2i}, -\frac{e^{\frac{i}{2}(z_1+z_2)} + e^{-\frac{i}{2}(z_1+z_2)}}{2} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.3) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, c_1 = 3\pi, c_2 = 2\pi$.

为叙述下述结果方便, 记 $s = \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}, s_0 = \frac{\alpha_1 c_2 - \alpha_2 c_1}{\alpha_1} \neq 0$, 令 $G_1(s), G_2(s)$ 是关于 s 且以 $2s_0$ 为周期的有限级超越整函数, 每次出现均可不同.

定理 2.3 若 $c = (c_1, c_2)$ 为 \mathbb{C}^2 上复常数, $\alpha_j, \beta_j (\neq 0) (j = 1, 2)$ 为 \mathbb{C} 上复常数, α_1, α_2 不同时为零. 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组

$$\begin{cases} \left(\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} \right)^2 + [\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2)]^2 = 1, \\ \left(\alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} \right)^2 + [\beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2)]^2 = 1 \end{cases} \quad (2.4)$$

的有限级超越整函数解, 那么 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有以下四种形式之一:

(i) 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有形式

$$(f_1, f_2) = (e^{\eta s} G_1(s) + A_0 s + \tau_1, e^{\eta s} G_2(s) + B_0 s + \tau_2),$$

这里 $\eta, \tau_1, \tau_2, A_0, B_0$ 为 \mathbb{C} 上复常数, 那么

(i₁) 当 $\beta_1 = \beta_2$ 时, 则 $\eta = 0, \tau_1 = \tau_2 = 0, A_0 = -B_0 = \frac{\alpha_1}{\beta_1} \frac{\xi_3 - \xi_2}{2(\alpha_1 c_2 - \alpha_2 c_1)}$, 以及

$$G_2(s + s_0) = -G_1(s) + \frac{\xi_2 + \xi_3}{2\beta_1};$$

(i₂) 当 $\beta_1 = -\beta_2$ 时, 则 $\eta = 0$, $\tau_1 = \tau_2 = 0$, $A_0 = B_0 = \frac{\alpha_1}{\beta_1} \frac{\xi_3 + \xi_2}{2(\alpha_1 c_2 - \alpha_2 c_1)}$, 以及

$$G_2(s + s_0) = G_1(s) + \frac{\xi_3 - \xi_2}{2\beta_1};$$

(i₃) 当 $\beta_1 \neq \pm\beta_2$ 时, 则 $A_0 = B_0 = 0$, $\eta = \alpha_1 \frac{\log \beta_2 - \log \beta_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $\tau_1 = \frac{\beta_1 \xi_3 - \beta_2 \xi_2}{\beta_1^2 - \beta_2^2}$, $\tau_2 = \frac{\beta_1 \xi_2 - \beta_2 \xi_3}{\beta_1^2 - \beta_2^2}$ 及

$$G_2(s + s_0) = -G_1(s),$$

这里 $\xi_2^2 = \xi_3^2 = 1$, $\xi_2, \xi_3 \in \mathbb{C}$;

(ii) 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有形式

$$(f_1, f_2) = \left(\xi_1 \frac{z_1}{\alpha_1} + A_0 s + G_1(s), -\xi_1 \frac{\beta_2}{\beta_1} \frac{z_1}{\alpha_1} + B_0 s + G_2(s) \right),$$

这里 $\xi_1^2 + \xi_2^2 = 1$, $\xi_2 = \pm \xi_3$, $A_0, B_0, \xi_1 (\neq 0)$, $\xi_2, \xi_3 \in \mathbb{C}$, 那么

(ii₁) 当 $\beta_1 = \beta_2$ 时, 则 $A_0 = -B_0 = \frac{1}{2} \left(\frac{\xi_3 - \xi_2}{\beta_1} - \frac{2c_1}{\alpha_1} \xi_1 \right) \cdot \frac{\alpha_1}{\alpha_1 c_2 - \alpha_2 c_1}$ 与

$$G_2(s + s_0) + G_1(s) = \frac{\xi_2 + \xi_3}{2\beta_1};$$

(ii₂) 当 $\beta_1 = -\beta_2$ 时, 则 $A_0 = B_0 = \frac{1}{2} \left(\frac{\xi_2 + \xi_3}{\beta_1} - \frac{2c_1}{\alpha_1} \xi_1 \right) \cdot \frac{\alpha_1}{\alpha_1 c_2 - \alpha_2 c_1}$ 与

$$G_2(s + s_0) = G_1(s) + \frac{\xi_2 - \xi_3}{2\beta_1};$$

(iii) 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有形式

$$(f_1, f_2) = \left(\frac{e^{L(z)+B_1} - e^{-(L(z)+B_1)}}{2(\alpha_1 a_1 + \alpha_2 a_2)} + e^{\eta s} G_1(s), \frac{e^{L(z)+B_2} - e^{-(L(z)+B_2)}}{2(\alpha_1 a_1 + \alpha_2 a_2)} + e^{\eta s} G_2(s) \right),$$

这里 $B_1, B_2, \eta \in \mathbb{C}$, 那么

$$e^{2L(c)} = 1, \quad \left(\frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} \right)^2 = 1, \quad e^{B_1 - B_2} = \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{L(c)}, \quad G_2(s + s_0) = K_0 G_1(s),$$

且 β_1, β_2, η 满足下列之一:

(iii₁) $\beta_1 \neq \pm\beta_2$, $\eta = \alpha_1 \frac{\log \beta_2 - \log \beta_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $K_0 = -1$;

(iii₂) $\beta_1 = \pm\beta_2$, $\eta = 0$, $K_0 = -\frac{\beta_2}{\beta_1}$;

(iv) 若 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 具有形式

$$(f_1, f_2) = \left(\frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2} \frac{z_1}{\alpha_1} + G_3(s), D_0 \frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2} \frac{z_1}{\alpha_1} + G_4(s) \right),$$

这里 $B_1, D_0 \in \mathbb{C}$, $\alpha_1 a_1 + \alpha_2 a_2 = 0$, 那么

(iv₁) 当 $e^{L(c)} = 1$ 时, $\beta_1 = \beta_2$, 则有 $D_0 = -1$,

$$G_3(s) = G_1(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\mu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_4(s) = G_2(s) + \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\nu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_2(s + s_0) = -G_1(s);$$

(iv₂) 当 $e^{L(c)} = 1$ 时, $\beta_1 = -\beta_2$, 则有 $D_0 = 1$,

$$G_3(s) = G_1(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\mu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_4(s) = G_2(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\nu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_2(s + s_0) = G_1(s);$$

(iv₃) 当 $e^{L(c)} = -1$ 时, $\beta_1 = \beta_2$, 则有 $D_0 = 1$,

$$G_3(s) = G_1(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\mu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_4(s) = G_2(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\nu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_2(s + s_0) = -G_1(s);$$

(iv₄) 当 $e^{L(c)} = -1$ 时, $\beta_1 = -\beta_2$, 则有 $D_0 = -1$,

$$G_3(s) = G_1(s) - \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\mu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_4(s) = G_2(s) + \frac{c_1}{2(\alpha_1 c_2 - \alpha_2 c_1)} s(e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\nu}{\beta_1} \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i},$$

$$G_2(s + s_0) = G_1(s),$$

其中 $\mu + \nu = 1$, $\mu, \nu \in \mathbb{C}$.

例 2.6 至例 2.16 说明了定理 2.3 各种情形解的存在性.

例 2.6 若

$$(f_1, f_2) = (e^{\pi i(z_2 - 2z_1)} + (z_2 - 2z_1), e^{\pi i(z_2 - 2z_1)} - (z_2 - 2z_1)),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 1$, $\beta_2 = 1$, $c_1 = 1$, $c_2 = 3$. 此例对应定理 2.3 (i₁) 中 $\xi_2 = -1$, $\xi_3 = 1$, $G_1(s) = G_2(s) = e^{\pi i s}$ 的情形.

例 2.7 若

$$(f_1, f_2) = \left(e^{-\pi i(z_2 - z_1)}, -e^{-\pi i(z_2 - z_1)} + \frac{1}{2} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 2$, $\beta_2 = -2$, $c_1 = 1$, $c_2 = 2$. 此例对应定理 2.3 (i₂) 中 $\xi_2 = -1$, $\xi_3 = 1$, $G_1(s) = -G_2(s) = e^{-\pi i s}$ 的情形.

例 2.8 若

$$(f_1, f_2) = \left(e^{\frac{\log 2}{2\pi i}(z_1 + z_2)} e^{z_1 + z_2} + \frac{1}{3}, e^{\frac{\log 2}{2\pi i}(z_1 + z_2)} e^{z_1 + z_2} + \frac{1}{3} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = \pi i$, $c_2 = \pi i$. 此例对应定理 2.3 (i₃) 中 $\xi_2 = 1$, $\xi_3 = 1$, $G_1(s) = G_2(s) = e^{s/2}$ 的情形.

例 2.9 若

$$(f_1, f_2) = \left(z_1 - z_2 + e^{\pi i(2z_2 - z_1)}, -z_1 + z_2 + e^{\pi i(2z_2 - z_1)} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 1$, $c_1 = 1$, $c_2 = 1$. 此例对应定理 2.3 (ii₁) 中 $\xi_1 = 1$, $\xi_2 = 0$, $\xi_3 = 0$, $G_1(s) = G_2(s) = e^{2\pi i s}$.

例 2.10 若

$$(f_1, f_2) = \left(\frac{z_2 - z_1}{2} + e^{z_2 - 2z_1}, \frac{z_2 - z_1}{2} - e^{z_2 - 2z_1} + \frac{\sqrt{3}}{2} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = 1, \beta_2 = -1, c_1 = \pi i, c_2 = \pi i$. 此例对应定理 2.3 (ii₂) 中 $\xi_1 = \frac{1}{2}, \xi_2 = \frac{\sqrt{3}}{2}, \xi_3 = -\frac{\sqrt{3}}{2}, G_1(s) = e^s, G_2(s) = -e^s + \frac{\sqrt{3}}{2}$.

例 2.11 若

$$(f_1, f_2) = \left(\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{4i} + e^{\frac{i}{4}(z_2-z_1)}, -\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{4i} - e^{\frac{i}{4}(z_2-z_1)} \right),$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 是方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = -1, \beta_2 = 1, c_1 = -\pi, c_2 = 3\pi$. 此例对应定理 2.3 (iii₁) 中 $a_1 = i, a_2 = i, G_1(s) = -G_2(s) = e^{\frac{i}{2}s}$.

例 2.12 若

$$\begin{aligned} f_1 &= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{6i} + e^{-\frac{\log 2}{4\pi}(2z_2-z_1)} e^{\frac{i}{4}(2z_2-z_1)}, \\ f_2 &= -\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{6i} + e^{-\frac{\log 2}{4\pi}(2z_2-z_1)} e^{\frac{i}{4}(2z_2-z_1)}, \end{aligned}$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2, c_1 = 2\pi, c_2 = -\pi$. 此例对应定理 2.3 (iii₂) 中 $a_1 = i, a_2 = i, G_1(s) = G_2(s) = e^{\frac{i}{2}s}$.

例 2.13 若 $a_1 = \frac{1}{2}, a_2 = -1, L(z) = z_2 - \frac{z_1}{2}$, 令

$$\begin{aligned} f_1 &= \frac{z_1 + 2z_2}{8}(e^{L(z)} + e^{-L(z)}) + \frac{1}{4i}(e^{L(z)} - e^{-L(z)}) + e^{z_2 - \frac{z_1}{2}}, \\ f_2 &= -\frac{z_1 + 2z_2}{8}(e^{L(z)} + e^{-L(z)}) + \frac{1}{4i}(e^{L(z)} - e^{-L(z)}) + e^{z_2 - \frac{z_1}{2}}, \end{aligned}$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 1, c_1 = 2\pi i, c_2 = -\pi i$. 此例对应定理 2.3 (iv₁) 中 $\mu = \nu = \frac{1}{2}, G_1(s) = G_2(s) = e^s$.

例 2.14 若 $a_1 = i, a_2 = -i, L(z) = i(z_1 - z_2)$, 令

$$\begin{aligned} f_1 &= \frac{z_1 + 3z_2}{8}(e^{L(z)} + e^{-L(z)}) - \frac{1}{6i}(e^{L(z)} - e^{-L(z)}) + e^{\frac{i}{4}(z_2-z_1)}, \\ f_2 &= \frac{z_1 + 2z_2}{8}(e^{L(z)} + e^{-L(z)}) + \frac{1}{6i}(e^{L(z)} - e^{-L(z)}) - e^{\frac{i}{4}(z_2-z_1)}, \end{aligned}$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \beta_2 = -1, c_1 = 3\pi, c_2 = -\pi$. 此例对应定理 2.3 (iv₂) 中 $\mu = \frac{1}{3}, \nu = \frac{2}{3}, G_1(s) = -G_2(s) = e^{\frac{i}{3}s}$.

例 2.15 若 $a_1 = 2, a_2 = -1, L(z) = 2z_1 - z_2$, 令

$$\begin{aligned} f_1 &= \frac{z_2 - z_1}{2}(e^{L(z)} + e^{-L(z)}) - \frac{1}{4}(e^{L(z)} - e^{-L(z)}) + e^{z_2 - 2z_1}, \\ f_2 &= \frac{z_2 - z_1}{2}(e^{L(z)} + e^{-L(z)}) + \frac{1}{4}(e^{L(z)} - e^{-L(z)}) + e^{z_2 - 2z_1}, \end{aligned}$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 1, \alpha_2 = 2, \beta_1 = i, \beta_2 = i, c_1 = \pi i, c_2 = \pi i$. 此例对应定理 2.3 (iv₃) 中 $\mu = \nu = \frac{1}{2}, G_1(s) = G_2(s) = e^s$.

例 2.16 若 $a_1 = 1, a_2 = 2, L(z) = z_1 + 2z_2$, 令

$$f_1 = \frac{z_1 + z_2}{2}(e^{L(z)} + e^{-L(z)}) + \frac{1}{2}(e^{L(z)} - e^{-L(z)}) + e^{\frac{1}{3}(z_1+2z_2)},$$

$$f_2 = -\frac{z_1 + z_2}{2}(e^{L(z)} + e^{-L(z)}) + e^{\frac{1}{3}(z_1 + 2z_2)},$$

则 $(f_1(z_1, z_2), f_2(z_1, z_2))$ 为方程组 (2.4) 的有限级超越整函数解, 其中 $\alpha_1 = 2, \alpha_2 = -1, \beta_1 = i, \beta_2 = -i, c_1 = -\pi i, c_2 = \pi i$. 此例对应定理 2.3 (iv₄) 中 $\mu = 1, \nu = 0, G_1(s) = -G_2(s) = e^{\frac{2}{3}s}$.

3 定理 2.1 的证明

设 (f_1, f_2) 是方程组 (2.1) 的有限级超越整函数解.

情形 1.1 $n_1 n_2 > m_1 m_2$. 由文 [4, 14] 知

$$m\left(r, \frac{f_j(z_1, z_2)}{f_j(z_1 + c_1, z_2 + c_2)}\right) = S(r, f_j), \quad j = 1, 2 \quad (3.1)$$

对所有的 $r > 0$ 成立, 至多除去一对数测度为无穷的例外集 $E_j \subset [1, +\infty)$, 即 $\int_{E_j} \frac{dt}{t} < \infty$. 于是

$$\begin{aligned} T(r, f_j(z_1, z_2)) &= m(r, f_j(z_1, z_2)) \\ &\leq m\left(r, \frac{f_j(z_1, z_2)}{f_j(z_1 + c_1, z_2 + c_2)}\right) + m(r, f_j(z_1 + c_1, z_2 + c_2)) + \log 2 \\ &= m(r, f_j(z_1 + c_1, z_2 + c_2)) + S(r, f_j) \\ &= T(r, f_j(z_1 + c_1, z_2 + c_2)) + S(r, f_j), \quad j = 1, 2, \end{aligned} \quad (3.2)$$

这里 $r \notin E =: E_1 \cup E_2$. 利用多变量对数导数引理 [1] 与 Mokhon'ko 定理 (见文 [11, 定理 3.4]), 由 (3.2) 知, 对所有的 $r \notin E$, 有

$$\begin{aligned} n_1 T(r, f_2) &\leq n_1 T(r, f_2(z_1 + c_1, z_2 + c_2)) + S(r, f_2) \\ &= T(r, f_2(z_1 + c_1, z_2 + c_2)^{n_1}) + S(r, f_2) \\ &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right)^{m_1} - 1\right) + S(r, f_2) \\ &= m_1 T\left(r, \alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right) + S(r, f_2) + S(r, f_1) \\ &= m_1 m\left(r, \alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right) + S(r, f_2) + S(r, f_1) \\ &\leq m_1 \left(m\left(r, \frac{\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}}{f_1}\right) + m(r, f_1)\right) + S(r, f_1) + S(r, f_2) \\ &= m_1 T(r, f_1) + S(r, f_1) + S(r, f_2). \end{aligned} \quad (3.3)$$

类似得

$$n_2 T(r, f_1(z_1, z_2)) \leq m_2 T(r, f_2(z_1, z_2)) + S(r, f_1) + S(r, f_2), \quad r \notin E. \quad (3.4)$$

由 (3.3) 与 (3.4), 得

$$(n_1 n_2 - m_1 m_2) T(r, f_j(z_1, z_2)) \leq S(r, f_1) + S(r, f_2), \quad r \notin E.$$

由于 $n_1 n_2 > m_1 m_2$, 上式与 f_1, f_2 为超越整函数矛盾.

情形 1.2 $m_j > \frac{n_j}{n_j - 1}, n_j \geq 2, j = 1, 2$. 于是

$$n_j > \frac{m_j}{m_j - 1}, \quad j = 1, 2.$$

一方面, 利用多变量 Nevanlinna 第二基本定理与差分对数导数引理 [4, 14], 有

$$\begin{aligned}
 & (m_1 - 1)T\left(r, \alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right) \\
 & \leq \overline{N}\left(r, \alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right) + \sum_{q=1}^{m_1} \overline{N}\left(r, \frac{1}{\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2} - w_q}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\
 & \leq \overline{N}\left(r, \frac{1}{(\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2})^{m_1} - 1}\right) + S\left(r, \frac{\partial f_1}{\partial z_1}\right) \\
 & \leq \overline{N}\left(r, \frac{1}{f_2(z_1 + c_1, z_2 + c_2)}\right) + S(r, f_1) \\
 & \leq T(r, f_2(z_1 + c_1, z_2 + c_2)) + S(r, f_1) + S(r, f_2),
 \end{aligned} \tag{3.5}$$

这里 w_q 为方程 $w^{m_1} - 1 = 0$ 的根. 类似地,

$$(m_2 - 1)T\left(r, \beta_1 \frac{\partial f_2}{\partial z_1} + \beta_2 \frac{\partial f_2}{\partial z_2}\right) \leq T(r, f_1(z_1 + c_1, z_2 + c_2)) + S(r, f_1) + S(r, f_2). \tag{3.6}$$

另一方面, 利用方程组 (2.1) 与多变量 Mokhon'ko 定理 (见文 [11, 定理 3.4]), 有

$$\begin{aligned}
 n_1 T(r, f_2(z_1 + c_1, z_2 + c_2)) &= T(r, f_2(z_1 + c_1, z_2 + c_2)^{n_1}) + S(r, f_2) \\
 &= T\left(r, \left(\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right)^{m_1} - 1\right) + S(r, f_2) \\
 &= m_1 T\left(r, \alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}\right) + S(r, f_1) + S(r, f_2).
 \end{aligned} \tag{3.7}$$

类似有

$$n_2 T(r, f_1(z_1 + c_1, z_2 + c_2)) = m_2 T\left(r, \beta_1 \frac{\partial f_2}{\partial z_1} + \beta_2 \frac{\partial f_2}{\partial z_2}\right) + S(r, f_1) + S(r, f_2). \tag{3.8}$$

由 (3.5)–(3.8), 得

$$\left(n_1 - \frac{m_1}{m_1 - 1}\right)T(r, f_2(z_1 + c_1, z_2 + c_2)) \leq S(r, f_1) + S(r, f_2),$$

与

$$\left(n_2 - \frac{m_2}{m_2 - 1}\right)T(r, f_1(z_1 + c_1, z_2 + c_2)) \leq S(r, f_1) + S(r, f_2),$$

由于 $n_j > \frac{m_j}{m_j - 1}$, 结合超越整函数 f_1, f_2 得到矛盾.

故定理 2.1 证毕.

4 定理 2.2 的证明

为证明定理 2.2 与定理 2.3, 需引入以下引理

引理 4.1 [12, 13] 若 $f_j (\neq 0)$, $j = 1, 2, 3$ 是 \mathbb{C}^m 上亚纯函数, f_1 非常数. 如果 $f_1 + f_2 + f_3 = 1$,

且

$$\sum_{j=1}^3 \left\{ N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j) \right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

对所有的 r 成立, 至多除去一对数测度为无穷的集, 这里 $0 < \lambda < 1$, 则 $f_2 = 1$ 或 $f_3 = 1$.

注 4.1 $N_2(r, \frac{1}{f})$ 为函数 f 在 $|z| \leq r$ 的零点计数函数, 其中简单零点计一次, 重级零点计两次.

引理 4.2 ^[23, 24] 设 F 是 \mathbb{C}^m 内整函数, $F(0) \neq 0$, 且 $\rho(n_F) = \rho < \infty$, 则存在一典型的函数 f_F 与 $g_F \in \mathbb{C}^m$ 满足 $F(z) = f_F(z)e^{g_F(z)}$. 特别地, 若 $m = 1$, f_F 为 Weierstrass 典型乘积.

注 4.2 记 $\rho(n_F)$ 为函数 F 的零点计数函数的级.

引理 4.3 ^[21] 若 g, h 为整函数, $g(h)$ 为有限级整函数, 那么以下两种情形之一发生:

- (a) h 为多项式, $\rho(g) < +\infty$;
- (b) h 不是多项式, $\rho(h) < +\infty$, $\rho(g) = 0$.

定理 2.2 的证明 设 (f_1, f_2) 是方程组 (2.3) 的一对有限级超越整函数解. 方程组 (2.3) 可表示为

$$\begin{cases} \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2} + i\alpha_3 f_2(z+c) \right] \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2} - i\alpha_3 f_2(z+c) \right] = 1, \\ \left[\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2} + i\alpha_3 f_1(z+c) \right] \left[\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2} - i\alpha_3 f_1(z+c) \right] = 1. \end{cases} \quad (4.1)$$

由于 f_1, f_2 为有限级超越整函数, 根据引理 4.2, 4.3 与 (4.1), 则存在两个非常数多项式 $p(z), q(z)$, 使得

$$\begin{cases} \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} + i\alpha_3 f_2(z_1 + c_1, z_2 + c_2) = e^{p(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} - i\alpha_3 f_2(z_1 + c_1, z_2 + c_2) = e^{-p(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} + i\alpha_3 f_1(z_1 + c_1, z_2 + c_2) = e^{q(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} - i\alpha_3 f_1(z_1 + c_1, z_2 + c_2) = e^{-q(z_1, z_2)}. \end{cases} \quad (4.2)$$

整理 (4.2) 得

$$\begin{cases} \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} = \frac{e^{p(z_1, z_2)} + e^{-p(z_1, z_2)}}{2}, \\ \alpha_3 f_2(z_1 + c_1, z_2 + c_2) = \frac{e^{p(z_1, z_2)} - e^{-p(z_1, z_2)}}{2i}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} = \frac{e^{q(z_1, z_2)} + e^{-q(z_1, z_2)}}{2}, \\ \alpha_3 f_1(z_1 + c_1, z_2 + c_2) = \frac{e^{q(z_1, z_2)} - e^{-q(z_1, z_2)}}{2i}, \end{cases} \quad (4.3)$$

即

$$\left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \right) e^{p(z+c)+q(z)} + \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \right) e^{p(z+c)-q(z)} - e^{2p(z+c)} \equiv 1, \quad (4.4)$$

$$\left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)+p(z)} + \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)-p(z)} - e^{2q(z+c)} \equiv 1. \quad (4.5)$$

显然, $\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \neq 0$. 否则, $e^{2q(z+c)} \equiv 1$, 与 $q(z)$ 为非常数多项式矛盾. 类似地, $\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \neq 0$. 于是, 根据引理 4.1, 再结合 (4.4) 与 (4.5), 得

$$\left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)-p(z)} \equiv 1 \quad \text{或} \quad \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)+p(z)} \equiv 1,$$

与

$$\left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2}\right) e^{p(z+c)-q(z)} \equiv 1 \quad \text{或} \quad \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2}\right) e^{p(z+c)+q(z)} \equiv 1.$$

下面分四种情形讨论.

情形 2.1

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2}\right) e^{q(z+c)-p(z)} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2}\right) e^{p(z+c)-q(z)} \equiv 1. \end{cases} \quad (4.6)$$

因为 $p(z), q(z)$ 是多项式, 由上式可得 $p(z+c)-q(z) \equiv \zeta_1, q(z+c)-p(z) \equiv \zeta_2$, 即 $p(z+2c)-p(z) \equiv \zeta_1-\zeta_2, q(z+2c)-q(z) \equiv \zeta_2-\zeta_1$, 此处及以下 ζ_1, ζ_2 为常数, 每次出现可不同. 于是 $p(z) = L(z) + H(z) + B_1, q(z) = L(z) + H(z) + B_2$, 这里 L 为线性函数 $L(z) = a_1 z_1 + a_2 z_2, a_1, a_2, B_1, B_2$ 为 \mathbb{C} 上复常数, $H(z) := H(s_1)$ 是 $s_1 = c_1 z_2 - c_2 z_1$ 的多项式. 结合 (4.4)–(4.6) 式, 有

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H'\right) e^{L(c)+B_2-B_1} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H'\right) e^{L(c)+B_1-B_2} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H'\right) e^{-L(c)-B_1+B_2} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H'\right) e^{-L(c)-B_2+B_1} \equiv 1, \end{cases} \quad (4.7)$$

这意味着

$$\begin{aligned} (\alpha_2 c_1 - \alpha_1 c_2) H' &\equiv 0, \quad \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2\right)^2 = 1, \\ e^{2L(c)} &= 1, \quad e^{B_1-B_2} = \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2\right) e^{L(c)}. \end{aligned} \quad (4.8)$$

情形 2.1.1 若 $e^{L(c)} = 1$, 则 $L(c) = 2k\pi i$ 以及 $e^{B_1-B_2} = \frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 = \frac{1}{(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2)}$. 由 (4.3) 得

$$\begin{aligned} f_1(z) &= \frac{e^{L(z)+H(z)+B_2-L(c)} - e^{-L(z)-H(z)-B_2+L(c)}}{2\alpha_3 i} \\ &= \frac{e^{L(z)+H(z)+B_2} - e^{-L(z)-H(z)-B_2}}{2\alpha_3 i} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2\alpha_3 i} \frac{1}{\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2(\alpha_1 a_1 + \alpha_2 a_2)} \end{aligned} \quad (4.9)$$

与

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+H(z)+B_1-L(c)} - e^{-L(z)-H(z)-B_1+L(c)}}{2\alpha_3 i} \\ &= \frac{-ie^{L(z)+H(z)+B_1} + ie^{-L(z)-H(z)-B_1}}{2\alpha_3}. \end{aligned} \quad (4.10)$$

情形 2.1.2 若 $e^{L(c)} = -1$, 那么 $L(c) = (2k+1)\pi i$, $e^{B_1-B_2} = -(\frac{\alpha_1}{\alpha_3 i}a_1 + \frac{\alpha_2}{\alpha_3 i}a_2) = -\frac{1}{\frac{\alpha_1}{\alpha_3 i}a_1 + \frac{\alpha_2}{\alpha_3 i}a_2}$. 结合 (4.3) 得

$$\begin{aligned} f_1(z) &= \frac{e^{L(z)+H(z)+B_2-L(c)} - e^{-L(z)-H(z)-B_2+L(c)}}{2\alpha_3 i} \\ &= -\frac{e^{L(z)+H(z)+B_2} - e^{-L(z)-H(z)-B_2}}{2\alpha_3 i} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2\alpha_3 i} \frac{1}{\frac{\alpha_1}{\alpha_3 i}a_1 + \frac{\alpha_2}{\alpha_3 i}a_2} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2(\alpha_1 a_1 + \alpha_2 a_2)} \end{aligned} \quad (4.11)$$

与

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+H(z)+B_1-L(c)} - e^{-L(z)-H(z)-B_1+L(c)}}{2\alpha_3 i} \\ &= \frac{ie^{L(z)+H(z)+B_1} - ie^{-L(z)-H(z)-B_1}}{2\alpha_3}. \end{aligned} \quad (4.12)$$

情形 2.2

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)-p(z)} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \right) e^{p(z+c)+q(z)} \equiv 1. \end{cases} \quad (4.13)$$

因为 $p(z), q(z)$ 是非常数多项式, 结合 (4.13) 得 $p(z+c) + q(z) \equiv \zeta_1$, $q(z+c) - p(z) \equiv \zeta_2$. 于是, $q(z+2c) + q(z) \equiv \zeta_1 + \zeta_2$, 这与 $q(z)$ 为非常数多项式矛盾.

情形 2.3

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)+p(z)} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \right) e^{p(z+c)-q(z)} \equiv 1. \end{cases} \quad (4.14)$$

因为 $p(z), q(z)$ 是非常数多项式, 结合 (4.14) 得 $p(z+c) - q(z) \equiv \zeta_1$, $q(z+c) + p(z) \equiv \zeta_2$. 于是 $p(z+2c) + p(z) \equiv \zeta_1 + \zeta_2$, 这与 $p(z)$ 为非常数多项式矛盾.

情形 2.4

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial p}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial p}{\partial z_2} \right) e^{q(z+c)+p(z)} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} \frac{\partial q}{\partial z_1} + \frac{\alpha_2}{\alpha_3 i} \frac{\partial q}{\partial z_2} \right) e^{p(z+c)+q(z)} \equiv 1. \end{cases} \quad (4.15)$$

因为 $p(z), q(z)$ 为非常数多项式, 结合 (4.15) 得 $p(z+c) + q(z) \equiv \zeta_1$, $q(z+c) + p(z) \equiv \zeta_2$, 即 $p(z+2c) - p(z) \equiv \zeta_1 + \zeta_2$, $q(z+2c) - q(z) \equiv \zeta_2 + \zeta_1$. 这样, $p(z) = L(z) + H(z) + B_1$, $q(z) = -L(z) - H(z) + B_2$, 这里 $L(z) = a_1 z_1 + a_2 z_2$, a_1, a_2, B_1, B_2 为 \mathbb{C} 上常数, $H(z)$ 如情形 2.1 中所

述. 结合 (4.4), (4.5) 和 (4.15) 式, 可得

$$\begin{cases} \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H' \right) e^{-L(c)+B_1+B_2} \equiv 1, \\ - \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H' \right) e^{L(c)+B_1+B_2} \equiv 1, \\ - \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H' \right) e^{-L(c)-B_1-B_2} \equiv 1, \\ \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 + \frac{\alpha_2 c_1 - \alpha_1 c_2}{\alpha_3 i} H' \right) e^{L(c)-B_1-B_2} \equiv 1, \end{cases} \quad (4.16)$$

于是

$$\begin{aligned} (\alpha_2 c_1 - \alpha_1 c_2) H' &\equiv 0, \quad \left(\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2 \right)^2 = 1, \\ e^{2L(c)} &= -1, \quad e^{B_1+B_2} = -\frac{1}{\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2} e^{-L(c)}. \end{aligned} \quad (4.17)$$

情形 2.4.1 若 $e^{L(c)} = i$, 则 $L(c) = (2k + \frac{1}{2})\pi i$, $e^{B_1+B_2} = \frac{i}{\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2}$. 结合 (4.3) 得

$$\begin{aligned} f_1(z) &= \frac{e^{-L(z)-H(z)+B_2+L(c)} - e^{L(z)+H(z)-B_2-L(c)}}{2\alpha_3 i} \\ &= \frac{e^{-L(z)-H(z)+B_2} + e^{L(z)+H(z)-B_2}}{2\alpha_3} \\ &= \frac{e^{L(z)+H(z)+B_1} e^{-B_1-B_2} + e^{-L(z)-H(z)-B_1} e^{B_1+B_2}}{2\alpha_3} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2(\alpha_1 a_1 + \alpha_2 a_2)} \end{aligned} \quad (4.18)$$

与

$$\begin{aligned} f_2(z) &= \frac{e^{L(z)+H(z)+B_1-L(c)} - e^{-L(z)-H(z)-B_1+L(c)}}{2\alpha_3 i} \\ &= \frac{-e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2\alpha_3}. \end{aligned} \quad (4.19)$$

情形 2.4.2 若 $e^{L(c)} = -i$, 则 $L(c) = (2k - \frac{1}{2})\pi i$, $e^{B_1+B_2} = -\frac{i}{\frac{\alpha_1}{\alpha_3 i} a_1 + \frac{\alpha_2}{\alpha_3 i} a_2}$. 结合 (4.3) 得

$$\begin{aligned} f_1(z) &= \frac{e^{-L(z)-H(z)+B_2+L(c)} - e^{L(z)+H(z)-B_2-L(c)}}{2\alpha_3 i} \\ &= \frac{-e^{L(z)+H(z)-B_2} - e^{-L(z)-H(z)+B_2}}{2\alpha_3} \\ &= \frac{-e^{L(z)+H(z)+B_1} e^{-B_1-B_2} - e^{-L(z)-H(z)-B_1} e^{B_1+B_2}}{2\alpha_3} \\ &= \frac{e^{L(z)+H(z)+B_1} - e^{-L(z)-H(z)-B_1}}{2(\alpha_1 a_1 + \alpha_2 a_2)} \end{aligned} \quad (4.20)$$

与

$$f_2(z) = \frac{e^{L(z)+H(z)+B_1-L(c)} - e^{-L(z)-H(z)-B_1+L(c)}}{2\alpha_3 i}$$

$$= \frac{e^{L(z)+H(z)+B_1} + e^{-L(z)-H(z)-B_1}}{2\alpha_3}. \quad (4.21)$$

综合情形 2.1 至情形 2.4, 定理 2.2 得证.

5 定理 2.3 的证明

引理 5.1 设 $c = (c_1, c_2)$ 为 \mathbb{C}^2 上复常数, $\alpha_j, (j = 1, 2)$ 为 \mathbb{C} 上复常数, α_1, α_2 不同时为零, $s_0 = \frac{\alpha_1 c_2 - \alpha_2 c_1}{\alpha_1} \neq 0$. 若方程

$$\alpha_1 \frac{\partial h}{\partial z_1} + \alpha_2 \frac{\partial h}{\partial z_2} = \gamma_0$$

的两个多项式解 $p(z), q(z)$ 满足 $q(z+c) - p(z) = \zeta_1, p(z+c) - q(z) = \zeta_2$, 其中 $\zeta_1, \zeta_2, \alpha_1, \alpha_2, \gamma_0$ 是 \mathbb{C} 上复常数, 则 $p(z) = L(z) + B_1, q(z) = L(z) + B_2$, 这里 $L(z) = a_1 z_1 + a_2 z_2, a_1, a_2, B_1, B_2 \in \mathbb{C}$.

证明 方程 $\alpha_1 \frac{\partial h}{\partial z_1} + \alpha_2 \frac{\partial h}{\partial z_2} = \gamma_0$ 的特征方程

$$\frac{dz_1}{dt} = \alpha_1, \quad \frac{dz_2}{dt} = \alpha_2, \quad \frac{dh}{dt} = \gamma_0,$$

取初值条件: $z_1 = 0, z_2 = s, h = h(0, s) := h_0(s)$. 于是 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$, 以及 $h = \int_0^t \gamma_0 dt + h_0(s) = \gamma_0 t + h_0(s)$, 这里 $s = \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}, h_0(s)$ 为 s 的一元函数. 由多项式 $p(z), q(z)$ 满足题设方程, 那么

$$p(z_1, z_2) = p(t, s) = \gamma_0 t + h_1(s), \quad q(z_1, z_2) = p(t, s) = \gamma_0 t + h_2(s), \quad (5.1)$$

这里 $h_1(s), h_2(s)$ 是关于 s 的两个多项式. 将 (5.1) 代入 $q(z+c) - p(z) = \zeta_1, q(z) - p(z+c) = \zeta_2$, 得

$$h_2(s + s_0) - h_1(s) = \zeta_1 - \frac{\gamma_0 c_1}{\alpha_1}, \quad h_1(s + s_0) - h_2(s) = \zeta_2 - \frac{\gamma_0 c_1}{\alpha_1}. \quad (5.2)$$

因此

$$h_1(s + 2s_0) = h_1(s) + \varepsilon_0, \quad h_2(s + 2s_0) = h_2(s) + \varepsilon_0, \quad (5.3)$$

其中 $\varepsilon_0 = \zeta_1 + \zeta_2 - 2\frac{\gamma_0 c_1}{\alpha_1}$. 由于 $h_1(s), h_2(s)$ 为 s 的多项式, 那么

$$h_1(s) = \gamma_1 s + B_1, \quad h_2(s) = \gamma_1 s + B_2, \quad (5.4)$$

其中 $\gamma_1 = \frac{\varepsilon_0}{2s_0}, B_1, B_2 \in \mathbb{C}$. 结合 (5.1) 与 (5.4), 整理可得

$$\begin{aligned} p(z_1, z_2) &= \gamma_0 t + \gamma_1 s + B_1 = \gamma_0 \frac{z_1}{\alpha_1} + \gamma_1 \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1} + B_1 = a_1 z_1 + a_2 z_2 + B_1, \\ q(z_1, z_2) &= \gamma_0 t + \gamma_1 s + B_2 = \gamma_0 \frac{z_1}{\alpha_1} + \gamma_1 \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1} + B_2 = a_1 z_1 + a_2 z_2 + B_2, \end{aligned}$$

其中 $a_1 = \frac{\gamma_0 - \gamma_1 \alpha_2}{\alpha_1}, a_2 = \gamma_1$. 于是, 引理 5.1 即证.

定理 2.3 的证明 设 (f_1, f_2) 是方程组 (2.4) 的一对超越有限整函数解. 由文 [7, 20] 知, Fermat 型方程 $f^2 + g^2 = 1$ 整函数解具有形式 $f = \cos a(z), g = \sin a(z)$, 其中 $a(z)$ 为整函数. 这样, f, g 要么为常数或要么为超越整函数.

(i) 若 $\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} = 0$. 由方程组 (2.4), 则

$$\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2) \equiv \xi_2, \quad \xi_2^2 = 1 \quad (5.5)$$

与

$$\alpha_1 \frac{\partial f_2(z+c)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z+c)}{\partial z_2} = -\frac{\beta_2}{\beta_1} \left(\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} \right) = 0. \quad (5.6)$$

再结合 (5.6) 与 (2.4), 易得

$$\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2} = 0, \quad \beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2) \equiv \xi_3, \quad \xi_3^2 = 1. \quad (5.7)$$

方程 $\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} = 0$ 的特征方程

$$\frac{dz_1}{dt} = \alpha_1, \quad \frac{dz_2}{dt} = \alpha_2, \quad \frac{df_1}{dt} = 0, \quad (5.8)$$

取初值条件 $z_1 = 0, z_2 = s, f_1 = f_1(0, s) := g_1(s)$. 根据特征方程 (5.8), 有 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$ 以及 $f_1 = \int_0^s 0 dt + g_1(s) = g_1(s)$, 这里 $s = \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}$, $g_1(s)$ 为有限级超越整函数. 于是

$$f_1(z_1, z_2) = g_1 \left(\frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1} \right). \quad (5.9)$$

类似可得

$$f_2(z_1, z_2) = g_2 \left(\frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1} \right), \quad (5.10)$$

其中 $g_2(s)$ 为有限级超越整函数.

将 (5.9), (5.10) 代入 (5.5), (5.7), 有

$$\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2) = \beta_1 g_2(s + s_0) + \beta_2 g_1(s) = \xi_2, \quad (5.11)$$

$$\beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2) = \beta_1 g_1(s + s_0) + \beta_2 g_2(s) = \xi_3. \quad (5.12)$$

(i₁) $\beta_1 = \beta_2$. 由 (5.11), (5.12) 得

$$g_1(s + 2s_0) = g_1(s) + \frac{\xi_3 - \xi_2}{\beta_1}, \quad g_2(s + 2s_0) = g_2(s) + \frac{\xi_2 - \xi_3}{\beta_1},$$

$$g_2(s + s_0) = -g_1(s) + \frac{\xi_2}{\beta_1}, \quad g_1(s + s_0) = -g_2(s) + \frac{\xi_3}{\beta_1}.$$

依据 $g_1(s), g_2(s)$ 是有限级超越整函数, 则

$$g_1(s) = G_1(s) + A_0 s, \quad g_2(s) = G_2(s) + B_0 s,$$

其中 $A_0 = -B_0 = \frac{\alpha_1}{\beta_1} \frac{\xi_3 - \xi_2}{2(\alpha_1 c_2 - \alpha_2 c_1)}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级超越整函数, 且

$$G_2(s + s_0) = -G_1(s) + \frac{\xi_2 + \xi_3}{2\beta_1}.$$

(i₂) $\beta_1 = -\beta_2$. 由 (5.11), (5.12) 得

$$g_1(s + 2s_0) = g_1(s) + \frac{\xi_2 + \xi_3}{\beta_1}, \quad g_2(s + 2s_0) = g_2(s) + \frac{\xi_2 + \xi_3}{\beta_1},$$

$$g_2(s + s_0) = g_1(s) + \frac{\xi_2}{\beta_1}, \quad g_1(s + s_0) = g_2(s) + \frac{\xi_3}{\beta_1}.$$

依据 $g_1(s), g_2(s)$ 是有限级超越整函数, 则

$$g_1(s) = G_1(s) + A_0 s, \quad g_2(s) = G_2(s) + B_0 s,$$

其中 $A_0 = B_0 = \frac{\alpha_1}{\beta_1} \frac{\xi_2 + \xi_3}{2(\alpha_1 c_2 - \alpha_2 c_1)}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级超越整函数, 且

$$G_2(s + s_0) = G_1(s) + \frac{\xi_2 - \xi_3}{2\beta_1}, \quad G_1(s + s_0) = G_2(s) + \frac{\xi_3 - \xi_2}{2\beta_1}.$$

(i₃) $\beta_1 \neq \pm\beta_2$. 由 (5.11), (5.12) 得

$$\begin{aligned} g_1(s + 2s_0) &= \left(\frac{\beta_2}{\beta_1}\right)^2 g_1(s) + \frac{-\beta_2\xi_2 + \beta_1\xi_3}{\beta_1}, \quad g_2(s + 2s_0) = \left(\frac{\beta_2}{\beta_1}\right)^2 g_2(s) + \frac{\beta_1\xi_2 - \beta_2\xi_3}{\beta_1}, \\ g_2(s + s_0) &= -\frac{\beta_2}{\beta_1}g_1(s) + \frac{\xi_2}{\beta_1}, \quad g_1(s + s_0) = -\frac{\beta_2}{\beta_1}g_2(s) + \frac{\xi_3}{\beta_1}. \end{aligned}$$

依据 $g_1(s), g_2(s)$ 是有限级超越整函数, 则

$$g_1(s) = e^{\eta s} G_1(s) + \tau_1, \quad g_2(s) = e^{\eta s} G_2(s) + \tau_2,$$

其中 $\eta = \alpha_1 \frac{\log \beta_2 - \log \beta_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $\tau_1 = \frac{\beta_1\xi_3 - \beta_2\xi_2}{\beta_1^2 - \beta_2^2}$, $\tau_2 = \frac{\beta_1\xi_2 - \beta_2\xi_3}{\beta_1^2 - \beta_2^2}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级超越整函数, 且 $G_2(s + s_0) = -G_1(s)$, $G_1(s + s_0) = -G_2(s)$.

(ii) 若

$$\alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} = \xi_1, \quad \xi_1 \neq 0. \quad (5.13)$$

类似于定理 2.3 (i) 中讨论可得

$$\alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} = -\frac{\beta_2}{\beta_1} \xi_1, \quad (5.14)$$

$$\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2) \equiv \xi_2, \quad \xi_1^2 + \xi_2^2 = 1, \quad (5.15)$$

$$\beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2) \equiv \xi_3, \quad \left(\frac{\beta_2}{\beta_1} \xi_1\right)^2 + \xi_3^2 = 1. \quad (5.16)$$

由 (5.13), (5.14) 和 (5.16), 得

$$\begin{aligned} \xi_1 &= \alpha_1 \frac{\partial f_1(z + c)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z + c)}{\partial z_2} \\ &= -\frac{\beta_2}{\beta_1} \left(\alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} \right) = \left(\frac{\beta_2}{\beta_1}\right)^2 \xi_1. \end{aligned}$$

又由 $\xi_1 \neq 0$, 则 $\beta_1^2 = \beta_2^2$, 即 $\beta_1 = \pm\beta_2$.

方程 (5.13) 的特征方程

$$\frac{dz_1}{dt} = \alpha_1, \quad \frac{dz_2}{dt} = \alpha_2, \quad \frac{df_1}{dt} = \xi_1, \quad (5.17)$$

取初值条件 $z_1 = 0, z_2 = s, f_1 = f_1(0, s) := g_1(s)$. 根据特征方程 (5.17), 有 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$ 以及 $f_1(t, s) = \int_0^t \xi_1 dt + g_1(s) = \xi_1 t + g_1(s)$, 这里 $s = \frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}$, $g_1(s)$ 为有限级超越整函数. 于是

$$f_1(z_1, z_2) = \xi_1 \frac{z_1}{\alpha_1} + g_1\left(\frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}\right). \quad (5.18)$$

上述方法应用于方程 (5.14), 得

$$f_2(z_1, z_2) = -\frac{\beta_2}{\beta_1} \xi_1 \frac{z_1}{\alpha_1} + g_2\left(\frac{\alpha_1 z_2 - \alpha_2 z_1}{\alpha_1}\right), \quad (5.19)$$

这里 $g_2(s)$ 为有限级超越整函数. 这样, 将 (5.18), (5.19) 代入 (5.15), (5.16), 整理可得

$$\beta_1 g_2(s + s_0) + \beta_2 g_1(s) = \frac{\beta_1 c_1}{\alpha_1} \xi_1 + \xi_2, \quad (5.20)$$

$$\beta_1 g_1(s+s_0) + \beta_2 g_2(s) = -\frac{\beta_1 c_1}{\alpha_1} \xi_1 + \xi_3. \quad (5.21)$$

(ii₁) $\beta_1 = \beta_2$. 由 (5.20), (5.21), 有

$$\begin{aligned} g_2(s+s_0) &= -g_1(s) + \frac{\xi_2}{\beta_1} + \frac{c_1}{\alpha_1} \xi_1, \quad g_1(s+s_0) = -g_2(s) + \frac{\xi_3}{\beta_1} - \frac{c_1}{\alpha_1} \xi_1, \\ g_1(s+2s_0) &= g_1(s) + \frac{\xi_3 - \xi_2}{\beta_1} - 2\frac{c_1}{\alpha_1} \xi_1, \quad g_2(s+2s_0) = g_2(s) + \frac{\xi_2 - \xi_3}{\beta_1} + 2\frac{c_1}{\alpha_1} \xi_1. \end{aligned}$$

由 $g_1(s), g_2(s)$ 是有限级超越整函数, 则

$$g_1(s) = G_1(s) + A_0 s, \quad g_2(s) = G_2(s) + B_0 s,$$

其中 $A_0 = -B_0 = \frac{1}{2}(\frac{\xi_3 - \xi_2}{\beta_1} - \frac{2c_1}{\alpha_1} \xi_1) \cdot \frac{\alpha_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级超越整函数, 且

$$G_2(s+s_0) + G_1(s) = \frac{\xi_2 + \xi_3}{2\beta_1}.$$

(ii₂) $\beta_1 = -\beta_2$. 由 (5.20), (5.21), 有

$$\begin{aligned} g_2(s+s_0) &= g_1(s) + \frac{\xi_2}{\beta_1} - \frac{c_1}{\alpha_1} \xi_1, \quad g_1(s+s_0) = g_2(s) + \frac{\xi_3}{\beta_1} - \frac{c_1}{\alpha_1} \xi_1, \\ g_1(s+2s_0) &= g_1(s) + \frac{\xi_3 + \xi_2}{\beta_1} - 2\frac{c_1}{\alpha_1} \xi_1, \quad g_2(s+2s_0) = g_2(s) + \frac{\xi_2 + \xi_3}{\beta_1} + 2\frac{c_1}{\alpha_1} \xi_1. \end{aligned}$$

由 $g_1(s), g_2(s)$ 是有限级超越整函数, 则

$$g_1(s) = G_1(s) + A_0 s, \quad g_2(s) = G_2(s) + B_0 s,$$

其中 $A_0 = B_0 = \frac{1}{2}(\frac{\xi_2 + \xi_3}{\beta_1} - \frac{2c_1}{\alpha_1} \xi_1) \cdot \frac{\alpha_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级超越整函数, 且

$$G_2(s+s_0) = G_1(s) + \frac{\xi_2 - \xi_3}{2\beta_1}.$$

(iii) 若 $\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2}$ 不是常数, 即为超越函数. 由方程组 (2.4) 第一式知 $\beta_1 f_2(z+c) + \beta_2 f_1(z)$ 为超越函数. 进而, $\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2}$ 与 $\beta_1 f_1(z+c) + \beta_2 f_2(z)$ 均为超越函数. 这样, 方程组 (2.4) 可写为

$$\begin{cases} \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2} + i(\beta_1 f_2(z+c) + \beta_2 f_1) \right] \left[\alpha_1 \frac{\partial f_1}{\partial z_1} + \alpha_2 \frac{\partial f_1}{\partial z_2} - i(\beta_1 f_2(z+c) + \beta_2 f_1) \right] = 1, \\ \left[\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2} + i(\beta_1 f_1(z+c) + \beta_2 f_2) \right] \left[\alpha_1 \frac{\partial f_2}{\partial z_1} + \alpha_2 \frac{\partial f_2}{\partial z_2} - i(\beta_1 f_1(z+c) + \beta_2 f_2) \right] = 1. \end{cases} \quad (5.22)$$

由于 f_1, f_2 为有限级超越整函数, 根据引理 4.2, 4.3 与 (5.22), 则存在两个非常数多项式 $p(z), q(z)$, 使得

$$\begin{cases} \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} + i[\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2)] = e^{p(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} - i[\beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2)] = e^{-p(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} + i[\beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2)] = e^{q(z_1, z_2)}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} - i[\beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2)] = e^{-q(z_1, z_2)}. \end{cases} \quad (5.23)$$

整理 (5.23) 得

$$\begin{cases} \alpha_1 \frac{\partial f_1(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_1(z_1, z_2)}{\partial z_2} = \frac{e^{p(z_1, z_2)} + e^{-p(z_1, z_2)}}{2}, \\ \beta_1 f_2(z_1 + c_1, z_2 + c_2) + \beta_2 f_1(z_1, z_2) = \frac{e^{p(z_1, z_2)} - e^{-p(z_1, z_2)}}{2i}, \\ \alpha_1 \frac{\partial f_2(z_1, z_2)}{\partial z_1} + \alpha_2 \frac{\partial f_2(z_1, z_2)}{\partial z_2} = \frac{e^{q(z_1, z_2)} + e^{-q(z_1, z_2)}}{2}, \\ \beta_1 f_1(z_1 + c_1, z_2 + c_2) + \beta_2 f_2(z_1, z_2) = \frac{e^{q(z_1, z_2)} - e^{-q(z_1, z_2)}}{2i}, \end{cases} \quad (5.24)$$

这意味着

$$\frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z)+q(z+c)} + \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z+c)-p(z)} - e^{2q(z+c)} \equiv 1, \quad (5.25)$$

$$\frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z)+p(z+c)} + \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z+c)-q(z)} - e^{2p(z+c)} \equiv 1. \quad (5.26)$$

显然, $\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} \neq \beta_2 i$. 否则, $-e^{2q(z+c)} \equiv 1$, 与 $q(z)$ 为非常数多项矛盾. 类似有 $\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} \neq \beta_2 i$. 于是, 应用引理 4.1 于 (5.25), (5.26), 得

$$\frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z+c)-p(z)} \equiv 1 \quad \text{或} \quad \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z)+q(z+c)} \equiv 1$$

与

$$\frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z+c)-q(z)} \equiv 1 \quad \text{或} \quad \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z)+p(z+c)} \equiv 1.$$

下面分四种情形讨论.

情形 3.1

$$\begin{cases} \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z+c)-p(z)} \equiv 1, \\ \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z+c)-q(z)} \equiv 1. \end{cases} \quad (5.27)$$

由于 $p(z), q(z)$ 是非常数多项式, 上式可得 $q(z+c) - p(z) \equiv \zeta_1$, $p(z+c) - q(z) \equiv \zeta_2$. 由引理 5.1 得 $p(z) = L(z) + B_1$, $q(z) = L(z) + B_2$, 这里 L 为线性函数 $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$, a_1, a_2 不同时为零. 结合 (5.25)–(5.27) 得

$$\begin{cases} \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{L(c)+B_2-B_1} \equiv 1, \\ \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{L(c)+B_1-B_2} \equiv 1, \\ \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{-L(c)+B_1-B_2} \equiv 1, \\ \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{-L(c)-B_1+B_2} \equiv 1. \end{cases} \quad (5.28)$$

于是

$$\left(\frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} \right)^2 = 1, \quad e^{2L(c)} = 1, \quad e^{B_1-B_2} = \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{L(c)}. \quad (5.29)$$

子情形 3.1.1 若 $\beta_1 \neq \pm\beta_2$. 由 (5.29) 有 $\alpha_1 a_1 + \alpha_2 a_2 = i(\beta_2 \pm \beta_1) \neq 0$. 方程组 (5.24) 第一式所对应的特征方程

$$\frac{dz_1}{dt} = \alpha_1, \quad \frac{dz_2}{dt} = \alpha_2, \quad \frac{df_1}{dt} = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2}, \quad (5.30)$$

取初值条件 $z_1 = 0, z_2 = s, f_1 = f_1(0, s) := g_0(s)$. 根据 (5.30), 则 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$ 以及

$$\begin{aligned} f_1(t, s) &= \int_0^t \frac{e^{(a_1 \alpha_1 + a_2 \alpha_2)t + a_2 s + B_1} + e^{-[(a_1 \alpha_1 + a_2 \alpha_2)t + a_2 s + B_1]}}{2} dt + g_0(s) \\ &= \frac{e^{a_2 s + B_1}}{2} \int_0^t e^{(a_1 \alpha_1 + a_2 \alpha_2)t} dt + \frac{e^{-(a_2 s + B_1)}}{2} \int_0^t e^{-(a_1 \alpha_1 + a_2 \alpha_2)t} dt + g_0(s) \\ &= \frac{e^{a_2 s + B_1}}{2(a_1 \alpha_1 + a_2 \alpha_2)} e^{(a_1 \alpha_1 + a_2 \alpha_2)t} - \frac{e^{-(a_2 s + B_1)}}{2(a_1 \alpha_1 + a_2 \alpha_2)} e^{-(a_1 \alpha_1 + a_2 \alpha_2)t} + g_1(s), \end{aligned}$$

这里 $g_1(s)$ 为有限级整函数, 且

$$g_1(s) = g_0(s) + \frac{e^{a_2 s + B_1}}{2(a_1 \alpha_1 + a_2 \alpha_2)} - \frac{e^{-(a_2 s + B_1)}}{2(a_1 \alpha_1 + a_2 \alpha_2)}.$$

结合 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$, 有

$$f_1(z_1, z_2) = \frac{e^{L(z)+B_1} - e^{-L(z)-B_1}}{2(a_1 \alpha_1 + a_2 \alpha_2)} + g_1(s). \quad (5.31)$$

类似地, 由方程组 (5.24) 第三式, 可得

$$f_2(z_1, z_2) = \frac{e^{L(z)+B_2} - e^{-L(z)-B_2}}{2(a_1 \alpha_1 + a_2 \alpha_2)} + g_2(s), \quad (5.32)$$

其中 $g_2(s)$ 为有限级整函数.

将 (5.31), (5.32) 代入方程组 (5.24) 中第二、四个方程, 结合 (5.29), 得

$$\beta_1 g_2(s + s_0) + \beta_2 g_1(s) = 0, \quad \beta_1 g_1(s + s_0) + \beta_2 g_2(s) = 0. \quad (5.33)$$

由 (5.33) 得

$$g_1(s) = e^{\eta s} G_1(s), \quad g_2(s) = e^{\eta s} G_2(s),$$

这里 $\eta = \alpha_1 \frac{\log \beta_2 - \log \beta_1}{\alpha_1 c_2 - \alpha_2 c_1}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且满足

$$G_2(s + s_0) = -G_1(s).$$

子情形 3.1.2 若 $\beta_1 = \pm\beta_2$. 由 (5.29) 有 $\alpha_1 a_1 + \alpha_2 a_2 = 2i\beta_2 \neq 0$ 或者 $\alpha_1 a_1 + \alpha_2 a_2 = 0$.

当 $\alpha_1 a_1 + \alpha_2 a_2 = 2i\beta_2 \neq 0$ 时, 类似子情形 3.1.1, 易得 (5.31)–(5.33). 由 (5.33) 得

$$g_1(s) = G_1(s), \quad g_2(s) = G_2(s),$$

这里 $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且 $G_2(s + s_0) = -\frac{\beta_2}{\beta_1} G_1(s)$.

当 $\alpha_1 a_1 + \alpha_2 a_2 = 0$ 时, 由 $z_1 = \alpha_1 t, z_2 = \alpha_2 t + s$, 结合 (5.42), 则 $L(z) = a_1 z_1 + a_2 z_2 = a_2 s$. 那么, 利用方程组 (5.24) 中第一式对应的特征方程 (5.30), 取初值条件 $z_1 = 0, z_2 = s, f_1 = f_1(0, s) := G_3(s)$, 得

$$\begin{aligned} f_1(t, s) &= \int_0^t \frac{e^{a_2 s + B_1} + e^{-(a_2 s + B_1)}}{2} dt + G_3(s) = \left(\frac{e^{a_2 s + B_1}}{2} + \frac{e^{-(a_2 s + B_1)}}{2} \right) t + G_3(s) \\ &= \frac{e^{a_2 s + B_1} + e^{-(a_2 s + B_1)}}{2} t + G_3(s), \end{aligned}$$

即

$$f_1(z_1, z_2) = \frac{e^{L(z)+B_1} + e^{-(L(z)+B_1)}}{2} \frac{z_1}{\alpha_1} + G_3(s), \quad (5.34)$$

这里 $G_3(s)$ 为有限级整函数. 类似有

$$f_2(z_1, z_2) = \frac{e^{L(z)+B_2} + e^{-L(z)-B_2}}{2} \frac{z_1}{\alpha_1} + G_4(s), \quad (5.35)$$

这里 $G_4(s)$ 为有限级整函数.

若 $e^{L(c)} = 1$, $\beta_1 = \beta_2$. 由 (5.29) 知 $e^{B_1-B_2} = -1$. 于是

$$f_2(z_1, z_2) = -\frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} \frac{z_1}{\alpha_1} + G_4(s).$$

将 (5.34), (5.35) 代入 (5.24) 中第二、四式, 得

$$\beta_1 G_4(s + s_0) + \beta_1 G_3(s) = \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} + \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}, \quad (5.36)$$

$$\beta_1 G_3(s + s_0) + \beta_1 G_4(s) = \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} - \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}. \quad (5.37)$$

于是

$$G_4(s + 2s_0) - G_4(s) = \frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}), \quad (5.38)$$

$$G_3(s + 2s_0) - G_3(s) = -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}). \quad (5.39)$$

由 (5.36)–(5.39), 得

$$G_3(s) = G_1(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\mu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}),$$

$$G_4(s) = G_2(s) + \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\nu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}),$$

其中 $\mu + \nu = 1$, $\mu, \nu \in \mathbb{C}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且 $G_2(s + s_0) = -G_1(s)$.

若 $e^{L(c)} = 1$, $\beta_1 = -\beta_2$. 由 (5.29) 知 $e^{B_1-B_2} = 1$. 于是

$$f_2(z_1, z_2) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} \frac{z_1}{\alpha_1} + G_4(s).$$

将 (5.34), (5.35) 代入 (5.24) 中第二、四式, 得

$$\beta_1 G_4(s + s_0) - \beta_1 G_3(s) = \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} - \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i},$$

$$\beta_1 G_3(s + s_0) - \beta_1 G_4(s) = -\frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} - \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}.$$

于是

$$G_4(s + 2s_0) - G_4(s) = -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}),$$

$$G_3(s + 2s_0) - G_3(s) = -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}).$$

这样, 上式可得

$$G_3(s) = G_1(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\mu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}),$$

$$G_4(s) = G_2(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\nu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}),$$

其中 $\mu + \nu = 1$, $\mu, \nu \in \mathbb{C}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且 $G_2(s + s_0) = G_1(s)$.

若 $e^{L(c)} = -1$, $\beta_1 = \beta_2$. 由 (5.29) 式知 $e^{B_1 - B_2} = 1$. 于是

$$f_2(z_1, z_2) = \frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} \frac{z_1}{\alpha_1} + G_4(s).$$

将 (5.34), (5.35) 代入 (5.24) 中第二、四式, 得

$$\begin{aligned} \beta_1 G_4(s + s_0) + \beta_1 G_3(s) &= \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} + \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}, \\ \beta_1 G_3(s + s_0) + \beta_1 G_4(s) &= -\frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} + \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}. \end{aligned}$$

于是

$$\begin{aligned} G_4(s + 2s_0) - G_4(s) &= -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}), \\ G_3(s + 2s_0) - G_3(s) &= -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}). \end{aligned}$$

上式可得

$$\begin{aligned} G_3(s) &= G_1(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) + \frac{\mu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}), \\ G_4(s) &= G_2(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\nu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}), \end{aligned}$$

其中 $\mu + \nu = 1$, $\mu, \nu \in \mathbb{C}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且 $G_2(s + s_0) = -G_1(s)$.

若 $e^{L(c)} = -1$, $\beta_1 = -\beta_2$. 由 (5.29) 知 $e^{B_1 + B_2} = -1$. 于是

$$f_2(z_1, z_2) = -\frac{e^{L(z)+B_1} + e^{-L(z)-B_1}}{2} \frac{z_1}{\alpha_1} + G_4(s).$$

将 (5.34), (5.35) 代入 (5.24) 中第二、四式, 得

$$\begin{aligned} \beta_1 G_4(s + s_0) - \beta_1 G_3(s) &= \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} - \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}, \\ \beta_1 G_3(s + s_0) - \beta_1 G_4(s) &= \frac{e^{a_2 s + B_1} - e^{-a_2 s - B_1}}{2i} + \beta_1 \frac{c_1}{\alpha_1} \frac{e^{a_2 s + B_1} + e^{-a_2 s - B_1}}{2i}. \end{aligned}$$

于是

$$\begin{aligned} G_4(s + 2s_0) - G_4(s) &= \frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}), \\ G_3(s + 2s_0) - G_3(s) &= -\frac{c_1}{\alpha_1} (e^{a_2 s + B_1} + e^{-a_2 s - B_1}). \end{aligned}$$

由上式有

$$\begin{aligned} G_3(s) &= G_1(s) - \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\mu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}), \\ G_4(s) &= G_2(s) + \frac{c_1}{2\alpha_1 s_0} s (e^{a_2 s + B_1} + e^{-a_2 s - B_1}) - \frac{\nu}{2\beta_1 i} (e^{a_2 s + B_1} - e^{-a_2 s - B_1}), \end{aligned}$$

其中 $\mu + \nu = 1$, $\mu, \nu \in \mathbb{C}$, $G_1(s), G_2(s)$ 是以 $2s_0$ 为周期的有限级整函数, 且 $G_2(s + s_0) = G_1(s)$.

情形 3.2

$$\begin{cases} \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z+c)-p(z)} \equiv 1, \\ \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z)+p(z+c)} \equiv 1. \end{cases}$$

因为 $p(z), q(z)$ 是非常数多项式, 由上述方程组得 $q(z+c) - p(z) \equiv \zeta_1$, $q(z) + p(z+c) \equiv \zeta_2$. 于是, $q(z+2c) + q(z) \equiv \zeta_1 + \zeta_2$, 与 $q(z)$ 是非常数多项式矛盾.

情形 3.3

$$\begin{cases} \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z)+q(z+c)} \equiv 1, \\ \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z+c)-q(z)} \equiv 1. \end{cases}$$

因为 $p(z), q(z)$ 是非常数多项式, 由上述方程组得 $p(z) + q(z+c) \equiv \zeta_1$, $p(z+c) - q(z) \equiv \zeta_2$. 于是 $p(z+2c) + p(z) \equiv \zeta_1 + \zeta_2$, 与 $p(z)$ 是非常数多项式矛盾.

情形 3.4

$$\begin{cases} \frac{\alpha_1 \frac{\partial p}{\partial z_1} + \alpha_2 \frac{\partial p}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{p(z)+q(z+c)} \equiv 1, \\ \frac{\alpha_1 \frac{\partial q}{\partial z_1} + \alpha_2 \frac{\partial q}{\partial z_2} - \beta_2 i}{\beta_1 i} e^{q(z)+p(z+c)} \equiv 1. \end{cases} \quad (5.40)$$

因为 $p(z), q(z)$ 是非常数多项式, 由 (5.40) 得 $p(z) + q(z+c) \equiv \zeta_1$, $q(z) + p(z+c) \equiv \zeta_2$. 于是 $p(z+2c) - p(z) \equiv \zeta_2 - \zeta_1$, $q(z+2c) - q(z) \equiv \zeta_1 - \zeta_2$. 由引理 5.1 得 $p(z) = L(z) + B_1$, $q(z) = -L(z) + B_2$, 这里 L 为线性函数 $L(z) = a_1 z_1 + a_2 z_2$, $a_1, a_2, B_1, B_2 \in \mathbb{C}$, a_1, a_2 不同时为零.

根据 (5.25), (5.26) 与 (5.40), 得

$$\begin{cases} \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{-L(c)+B_1+B_2} \equiv 1, \\ \frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} e^{L(c)-B_1-B_2} \equiv 1, \\ \frac{-(\alpha_1 a_1 + \alpha_2 a_2) - \beta_2 i}{\beta_1 i} e^{L(c)+B_1+B_2} \equiv 1, \\ \frac{-(\alpha_1 a_1 + \alpha_2 a_2) - \beta_2 i}{\beta_1 i} e^{-L(c)-B_1-B_2} \equiv 1, \end{cases} \quad (5.41)$$

上式得

$$\left(\frac{\alpha_1 a_1 + \alpha_2 a_2 - \beta_2 i}{\beta_1 i} \right)^2 = \left(\frac{\alpha_1 a_1 + \alpha_2 a_2 + \beta_2 i}{\beta_1 i} \right)^2,$$

于是

$$\alpha_1 a_1 + \alpha_2 a_2 = 0, \text{ 或 } \beta_2 = 0 \text{ (与题设矛盾)}. \quad (5.42)$$

结合 (5.41) 与 (5.42), 得

$$\beta_1 = \pm \beta_2, \quad e^{2L(c)} = 1, \quad e^{B_1+B_2} = -\frac{\beta_2}{\beta_1} e^{L(c)}. \quad (5.43)$$

类似于子情形 3.1.2 的讨论可得定理 2.3 (iv) 的结论.

综上, 定理 2.3 证毕.

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