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# Banach 空间中渐近非扩张映射的广义粘性隐式双中点法则

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**摘　要**　本文给出了实 Banach 空间中, 渐近非扩张映射不动点的广义隐式双中点法则的粘性方法. 在适当的参数条件下, 证明了该算法生成的序列的强收敛定理. 本文的结果推广和改进了其他作者的主要结果.

**关键词**　Banach 空间; 渐近非扩张映射; 广义粘性隐式双中点法则; 强收敛
**MR(2010) 主题分类**　47H09, 47H10
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## The Generalized Viscosity Implicit Double Midpoint Rule for Asymptotically Non-expansive Mappings in Banach Spaces

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**Abstract**　We study viscosity method for the general implicit double midpoint rule for finding the fixed points of asymptotically non-expansive mappings in real Banach spaces. Under suitable conditions imposed on the parameters, some strong convergence theorems of the sequence generated by the algorithm are proved. The results presented in this article extend and improve the main results of other authors.

**Keywords**　Banach space; asymptotically non-expansive mapping; generalized viscosity implicit double midpoint rule; strong convergence
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## 1　引言

我们假设 $X$ 是一个 Banach 空间, $X^*$ 是 $X$ 的对偶空间, 对偶映射 $J: X \to 2^{X^*}$ 定义为

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\,\|f\|, \|x\| = \|f\|\},\quad \forall x \in X,$$

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其中 $\langle\cdot,\cdot\rangle$ 表示对偶配对. 众所周知, 算子 $J$ 是有意义的且在 Hilbert 空间中 $J$ 是恒等算子, 但一般情况下, $J$ 是多值的且非线性的. 如果对于 $\|x\|=\|y\|=1$ 和 $x\neq y$, 有 $\frac{\|x+y\|}{2}<1$, 称 $X$ 为严格凸的. $X$ 的凸性模定义为

$$\delta_X(\epsilon)=\inf\left\{1-\left\|\frac{1}{2}(x+y)\right\|:\|x\|,\ \|y\|\leq 1,\ \|x-y\|\geq\epsilon\right\}$$

对所有的 $0\leq\epsilon\leq 2$ 都成立. 如果 $\delta_X(\epsilon)>0$ 对所有的 $0<\epsilon\leq 2$ 都成立, $X$ 被称为一致凸. 设 $\rho_X:[0,+\infty)\to[0,+\infty)$ 是 $X$ 的光滑模, 定义为

$$\rho_X(s)=\sup\left\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1:\|x\|=1,\ \|y\|\leq s\right\}.$$

当 $s\to 0$ 时, $\frac{\rho_X(s)}{s}\to 0$, $X$ 称为一致光滑的. 如果 $X$ 是一致 $F$ 可微的, 则 $X$ 也是一致光滑的.

设 $C$ 是实 Banach 空间 $X$ 的非空闭凸子集, $T:C\to C$ 是一个映射, $T$ 的不动点集合表示为 $F(T)$. 映射 $T:C\to C$ 称为渐近非扩张的, 如果存在数列 $\{k_n\}\subset[1,\infty)$, $\lim_{n\to\infty}k_n=1$,使得

$$\|T^nx-T^ny\|\leq k_n\|x-y\|,\quad\forall\,x,y\in C,\quad\forall\,n\geq 1.$$

$T$ 是一致渐近正则的当且仅当 $\lim_{n\to\infty}\|T^{n+1}x-T^nx\|=0,\ \forall\,x\in C$. 如果 $k_n=1$, 那么 $T$ 称为非扩张的. 映射 $T$ 称为 $\alpha$ 压缩的, 如果有

$$\|T(x)-T(y)\|\leq\alpha\|x-y\|,\quad\forall\,x,y\in X,$$

其中常数 $\alpha\in(0,1)$.

变分不等式理论和不动点理论是非线性分析和优化问题的两个重要领域, 在 Hilbert 空间或 Banach 空间中, 由于它们在信号处理, 鞍点问题, 平衡问题和博弈论中的广泛应用, 把这些问题转化为不动点问题已经受到了很大关注 [3, 6, 9, 14–16, 18, 21]. 隐式中点法则是求解某些微分代数方程的重要数值方法之一, 利用隐式中点法则进行粘性迭代算法的收敛性分析已有许多介绍, 详见文献 [1, 4, 5, 7, 8, 11, 13, 17, 19, 20].

2015 年, Xu [17] 在 Hilbert 空间中介绍了非扩张映射的粘性隐式中点法则, 迭代序列如下:

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)T\left(\frac{x_n+x_{n+1}}{2}\right),\quad\forall\,n\geq 0,$$

并证明了该序列强收敛于 $T$ 的不动点 $x^*$, 这也是下面变分不等式问题的解,

$$\langle(I-f)x^*,x-x^*\rangle\geq 0,\quad\forall\,x\in F(T).\tag{1.1}$$

2019 年, Pan [11] 在 Banach 空间中研究了渐近非扩张映射广义粘性隐式法则, 如下:

$$x_{n+1}=\alpha_nx_n+\beta_nf(x_n)+\gamma_nT^n(t_nx_n+(1-t_n)x_{n+1}),\quad\forall\,n\geq 0,$$

并证明了上式生成的迭代序列 $x_n$ 强收敛于 $x^*\in F(T)$, 这也是变分不等式问题 (1.1) 的解.

最近, Dhakal [5] 在 Hilbert 空间中研究了关于非扩张映射不动点的隐式双中点法则的粘性方法, 如下:

$$x_{n+1}=\alpha_nf\left(\frac{x_n+x_{n+1}}{2}\right)+(1-\alpha_n)T\left(\frac{x_n+x_{n+1}}{2}\right),\quad\forall\,n\geq 0,\tag{1.2}$$

并证明了由 (1.2) 式生成的迭代序列 $x_n$ 强收敛于 $x^*\in F(T)$, 这也是变分不等式问题 (1.1) 的解.

基于以上结果, 我们在 Banach 空间中提出关于渐近非扩张映射的广义粘性隐式双中点法则, 在适当的参数条件下, 推广了 Dhakal [5] 的主要结果, 分别从 Hilbert 空间推广到 Banach 空间,

非扩张映射推广到渐近非扩张映射和粘性隐式中点法则推广到更广义粘性隐式中点法则, 进而也推广或部分推广了许多作者的主要结果 [2, 5, 7, 10–12, 17].

## 2 预备知识

为了证明本文的主要结果, 还需要下面的重要引理.

**引理 2.1** [11] 设 $\{x_n\},\{z_n\}$ 是 Banach 空间 $X$ 上的两个有界序列, $\mu_n$ 是 $[0,1]$ 中的一实数列, 且满足 $0<\liminf_{n\to\infty}\mu_n\le\limsup_{n\to\infty}\mu_n<1$. 若对所有的 $n\ge 0$, 满足 $x_{n+1}=(1-\mu_n)x_n+\mu_n z_n$ 和 $\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0$, 则 $\lim_{n\to\infty}\|z_n-x_n\|=0$.

**引理 2.2** [12] 设 $X$ 是带有弱序列连续对偶映射 $J$ 的实 Banach 空间, $\emptyset\ne C\subset X$ 是一个有界闭凸子集. 假设 $T:C\to C$ 是渐近非扩张映射, 则 $I-T$ 在零点是半闭的, 其中 $I$ 是恒等映射. 即如果 $x_n\rightharpoonup x$, $\|x_n-Tx_n\|\to 0$, 则 $x\in F(T)$.

**引理 2.3** [10] 设 $X$ 是实 Banach 空间, $\emptyset\ne C\subset X$ 是一个有界闭凸子集. 假设 $T:C\to C$ 是非扩张映射且 $F(T)\ne\emptyset$, $f:C\to C$ 是压缩映射, 定义为 $x_v=vf(x_v)+(1-v)Tx$, $v\in(0,1)$ 的序列 $\{x_v\}$ 在 $F(T)$ 上强收敛于一点, 假设 $Q:\Pi_C\to F(T)$, 且 $Q(f)=\lim_{v\to 0}x_v, f\in\Pi_C$, 则 $Q(f)$ 是下面不等式的解:

$$\langle (I-f)Q(f),j(Q(f)-\widehat{x})\rangle\le 0,\quad \forall\widehat{x}\in F(T).$$

**引理 2.4** [2] 设 $\{c_n\}$ 是一非负实数列, 使得 $c_{n+1}\le(1-\sigma_n)c_n+\sigma_n\theta_n$, $\forall n\ge 0$, 其中 $\{\sigma_n\},\{\theta_n\}$ 满足:

(i) $\{\sigma_n\}\subset[0,1]$, $\sum_{n=0}^{\infty}\sigma_n=\infty$;

(ii) $\limsup_{n\to\infty}\theta_n\le 0$ 或 $\sum_{n=1}^{\infty}|\sigma_n\theta_n|<\infty$,

则 $\lim_{n\to\infty}c_n=0$.

## 3 主要定理及其证明

**定理 3.1** 设 $X$ 是 2 一致凸和一致光滑的 Banach 空间, $C\subset X$ 是非空有界闭凸集, $T:C\to C$ 是渐近非扩张映射且 $F(T)\ne\emptyset$, $f$ 是 $C$ 上的 $\gamma\in[0,1)$ 压缩映射. 若 $x_0\in C$, 在 $C$ 上定义如下迭代序列 $\{x_n\}$:

$$x_{n+1}=\mu_n x_n+\nu_n f(t_n x_n+(1-t_n)x_{n+1})+\omega_n T^n(t_n x_n+(1-t_n)x_{n+1}),\quad n\ge 0,\tag{3.1}$$

其中对于所有的 $n\in N$, $\mu_n+\nu_n+\omega_n=1$ 且满足以下条件:

(i) $0\le\mu_n,\nu_n,\omega_n<1$, $\lim_{n\to\infty}\nu_n=0$, $\sum_{n=0}^{\infty}\nu_n=\infty$, $k_n-1=\varepsilon\nu_n$, $\varepsilon<1-\gamma$;

(ii) $\lim_{n\to\infty}|\mu_{n+1}-\mu_n|=0$, $\lim_{n\to\infty}|\nu_{n+1}-\nu_n|=0$, $\lim_{n\to\infty}|\omega_{n+1}-\omega_n|=0$;

(iii) $0<\liminf_{n\to\infty}t_n\le\limsup_{n\to\infty}t_{n+1}<1$, $(\nu_n\gamma+\omega_n k_n)(1-t_n)<1$, $\forall n>0$,

则 $\{x_n\}$ 强收敛于不动点 $x^*\in F(T)$, 这也是变分不等式问题 (1.1) 的解.

**证明 第 1 步** 假设 $q\in F(T)$, 对于 $n\in\mathbb{N}$, 我们有

$$\begin{aligned}\|x_{n+1}-q\|&=\|\mu_n x_n+\nu_n f(t_n x_n+(1-t_n)x_{n+1})+\omega_n T^n(t_n x_n+(1-t_n)x_{n+1})-q\|\\&\le\|\mu_n(x_n-q)+\nu_n(f(t_n x_n+(1-t_n)x_{n+1})-q)+\omega_n(T^n(t_n x_n+(1-t_n)x_{n+1})-q)\|\\&\le\mu_n\|x_n-q\|+\nu_n\|f(t_n x_n+(1-t_n)x_{n+1})-f(q)\|+\nu_n\|f(q)-q\|\\&\quad+\omega_n\|T^n(t_n x_n+(1-t_n)x_{n+1})-q\|\end{aligned}$$

$$
\begin{aligned}
&\le \mu_n\|x_n-q\|+\nu_n\gamma\|(t_nx_n+(1-t_n)x_{n+1})-q\|+\nu_n\|f(q)-q\|\\
&\quad+\omega_nk_n\|(t_nx_n+(1-t_n)x_{n+1})-q\|\\
&\le \mu_n\|x_n-q\|+\nu_n\gamma\|t_n(x_n-q)+(1-t_n)(x_{n+1}-q)\|+\nu_n\|f(q)-q\|\\
&\quad+\omega_nk_n\|t_n(x_n-q)+(1-t_n)(x_{n+1}-q)\|\\
&\le \mu_n\|x_n-q\|+\nu_n\gamma t_n\|x_n-q\|+\nu_n\gamma(1-t_n)\|x_{n+1}-q\|+\nu_n\|f(q)-q\|\\
&\quad+\omega_nk_nt_n\|x_n-q\|+\omega_nk_n(1-t_n)\|x_{n+1}-q\|\\
&\le [\mu_n+(\nu_n\gamma+\omega_nk_n)t_n]\|x_n-q\|+(\nu_n\gamma+\omega_nk_n)(1-t_n)\|x_{n+1}-q\|+\nu_n\|f(q)-q\|,
\end{aligned}
$$

即

$$[1-(\nu_n\gamma+\omega_nk_n)(1-t_n)]\|x_{n+1}-q\|\le(\mu_n+(\nu_n\gamma+\omega_nk_n)t_n)\|x_n-q\|+\nu_n\|f(q)-q\|.$$

根据条件 (i), (iii), 得到

$$
\begin{aligned}
\|x_{n+1}-q\|&\le\frac{\mu_n+(\nu_n\gamma+\omega_nk_n)t_n}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\|x_n-q\|+\frac{\nu_n}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\|f(q)-q\|\\
&\le\left(1-\frac{\nu_n(1-\gamma)-\omega_n(k_n-1)}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\right)\|x_n-q\|+\frac{\nu_n}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\|f(q)-q\|\\
&\le\left(1-\frac{\nu_n(1-\gamma)-\varepsilon\nu_n}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\right)\|x_n-q\|+\frac{\nu_n}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\|f(q)-q\|\\
&\le\left(1-\frac{\nu_n(1-\gamma-\varepsilon)}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\right)\|x_n-q\|+\frac{\nu_n(1-\gamma-\varepsilon)}{1-(\nu_n\gamma+\omega_nk_n)(1-t_n)}\frac{\|f(q)-q\|}{1-\gamma-\varepsilon}\\
&\le\max\left\{\|x_n-q\|,\frac{\|f(q)-q\|}{1-\gamma-\varepsilon}\right\}.
\end{aligned}
$$

由数学归纳法得到

$$\|x_{n+1}-q\|\le\max\left\{\|x_1-q\|,\frac{\|f(q)-q\|}{1-\gamma-\varepsilon}\right\}.$$

因此 $\{x_n\}$ 是有界的, 进而 $\{f(t_nx_n+(1-t_n)x_{n+1})\},\{T^n(t_nx_n+(1-t_n)x_{n+1})\}$ 也有界.

**第 2 步** 由 $x_n$ 的定义, 我们知道

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\|&=\|\mu_{n+1}x_{n+1}+\nu_{n+1}f(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})\\
&\quad+\omega_{n+1}T^{n+1}(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})\\
&\quad-[\mu_nx_n+\nu_nf(t_nx_n+(1-t_n)x_{n+1})+\omega_nT^n(t_nx_n+(1-t_n)x_{n+1})]\|\\
&\le\|\mu_{n+1}(x_{n+1}-x_n)+(\mu_{n+1}-\mu_n)x_n+\nu_{n+1}[f(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})\\
&\quad-f(t_nx_n+(1-t_n)x_{n+1})]+(\nu_{n+1}-\nu_n)f(t_nx_n+(1-t_n)x_{n+1})\\
&\quad+\omega_{n+1}[T^{n+1}(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})-T^{n+1}(t_nx_n+(1-t_n)x_{n+1})]\\
&\quad+(\omega_{n+1}-\omega_n)T^{n+1}(t_nx_n+(1-t_n)x_{n+1})\\
&\quad+\omega_n[T^{n+1}(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})]\|\\
&\le\mu_{n+1}\|x_{n+1}-x_n\|+|\mu_{n+1}-\mu_n|\,\|x_n\|+\nu_{n+1}\|f(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})\\
&\quad-f(t_nx_n+(1-t_n)x_{n+1})\|+|\nu_{n+1}-\nu_n|\,\|f(t_nx_n+(1-t_n)x_{n+1})\|\\
&\quad+\omega_{n+1}\|T^{n+1}(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})-T^{n+1}(t_nx_n+(1-t_n)x_{n+1})\|\\
&\quad+|\omega_{n+1}-\omega_n|\,\|T^{n+1}(t_nx_n+(1-t_n)x_{n+1})\|\\
&\quad+\omega_n\|T^{n+1}(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \mu_{n+1}\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\,\|x_n\| + \nu_{n+1}\gamma\|(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})\\
&\quad -(t_nx_n+(1-t_n)x_{n+1})\| + |\nu_{n+1}-\nu_n|\,\|f(x)\|\\
&\quad +\omega_{n+1}k_{n+1}\|(t_{n+1}x_{n+1}+(1-t_{n+1})x_{n+2})-(t_nx_n+(1-t_n)x_{n+1})\|\\
&\quad +|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|+\omega_n\|T^{n+1}x-T^nx\|\\
&\leq \mu_{n+1}\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\,\|x_n\| + \nu_{n+1}\gamma\|t_n(x_{n+1}-x_n)\\
&\quad +(1-t_{n+1})(x_{n+2}-x_{n+1})\| + |\nu_{n+1}-\nu_n|\,\|f(x)\|\\
&\quad +\omega_{n+1}k_{n+1}\|t_n(x_{n+1}-x_n)+(1-t_{n+1})(x_{n+2}-x_{n+1})\|\\
&\quad +|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|+\omega_n\|T^{n+1}x-T^nx\|\\
&\leq \mu_{n+1}\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\,\|x_n\| + \nu_{n+1}\gamma t_n\|x_{n+1}-x_n\|\\
&\quad +\nu_{n+1}\gamma(1-t_{n+1})\|x_{n+2}-x_{n+1}\| + |\nu_{n+1}-\nu_n|\,\|f(x)\|\\
&\quad +\omega_{n+1}k_{n+1}t_n\|x_{n+1}-x_n\|+\omega_{n+1}k_{n+1}(1-t_{n+1})\|x_{n+2}-x_{n+1}\|\\
&\quad +|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|+\omega_n\|T^{n+1}x-T^nx\|\\
&\leq [\mu_{n+1}+(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n]\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\,\|x_n\|\\
&\quad +(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})]\|x_{n+2}-x_{n+1}\|\\
&\quad +|\nu_{n+1}-\nu_n|\,\|f(x)\|+\omega_n\|T^{n+1}x-T^nx\|+|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|,
\end{aligned}$$

即

$$\begin{aligned}
&[1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})]\|x_{n+2}-x_{n+1}\|\\
&\quad\leq [\mu_{n+1}+(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n]\|x_{n+1}-x_n\| + |\mu_{n+1}-\mu_n|\,\|x_n\|\\
&\qquad +|\nu_{n+1}-\nu_n|\,\|f(x)\|+\omega_n\|T^{n+1}x-T^nx\|+|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|.
\end{aligned}$$

因此

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \frac{\mu_{n+1}+(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\|x_{n+1}-x_n\|\\
&\quad+\frac{|\mu_{n+1}-\mu_n|\,\|x_n\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}+\frac{|\nu_{n+1}-\nu_n|\,\|f(x)\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\\
&\quad+\frac{|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\\
&\quad+\frac{\omega_n}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\sup_{x\in X}\|T^{n+1}x-T^nx\|\\
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma)-\omega_{n+1}(k_{n+1}-1)}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\right]\|x_{n+1}-x_n\|\\
&\quad+\frac{|\mu_{n+1}-\mu_n|\,\|x_n\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}+\frac{|\nu_{n+1}-\nu_n|\,\|f(x)\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\\
&\quad+\frac{|\omega_{n+1}-\omega_n|\,\|T^{n+1}x\|}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\\
&\quad+\frac{\omega_n}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\sup_{x\in X}\|T^{n+1}x-T^nx\|\\
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma)-\varepsilon\nu_{n+1}}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\right]\|x_{n+1}-x_n\|+M_{n_1}\\
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma-\varepsilon)}{1-(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}\right]\|x_{n+1}-x_n\|+M_{n_1}\\
&\leq \|x_{n+1}-x_n\|+M_{n_1},
\end{aligned}\tag{3.2}$$

其中

$$M_{n_1} = \frac{|\mu_{n+1} - \mu_n| \, \|x_n\|}{1 - (\nu_{n+1}\gamma + \omega_{n+1}k_{n+1})(1 - t_{n+1})} + \frac{|\nu_{n+1} - \nu_n| \, \|f(x)\|}{1 - (\nu_{n+1}\gamma + \omega_{n+1}k_{n+1})(1 - t_{n+1})} + \frac{|\omega_{n+1} - \omega_n| \, \|T^{n+1}x\|}{1 - (\nu_{n+1}\gamma + \omega_{n+1}k_{n+1})(1 - t_{n+1})} + \frac{\omega_n \sup_{x \in X} \|T^{n+1}x - T^n x\|}{1 - (\nu_{n+1}\gamma + \omega_{n+1}k_{n+1})(1 - t_{n+1})}.$$

设 $z_n = \frac{x_{n+1} - \mu_n x_n}{1 - \mu_n}$, $\forall n \geq 0$. 因为

$$x_{n+1} - \mu_n x_n = \nu_n f(t_n x_n + (1 - t_n)x_{n+1}) + \omega_n T^n(t_n x_n + (1 - t_n)x_{n+1}),$$

所以

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \mu_{n+1}x_{n+1}}{1 - \mu_{n+1}} - \frac{x_{n+1} - \mu_n x_n}{1 - \mu_n} \\
&= \frac{\nu_{n+1} f(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2}) + \omega_{n+1}T^{n+1}(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2})}{1 - \mu_{n+1}} \\
&\quad - \frac{\nu_n f(t_n x_n + (1 - t_n)x_{n+1}) + \omega_n T^n(t_n x_n + (1 - t_n)x_{n+1})}{1 - \mu_n} \\
&\leq \frac{\nu_{n+1}}{1 - \mu_{n+1}}[f(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2}) - f(t_n x_n + (1 - t_n)x_{n+1})] \\
&\quad + \left(\frac{\nu_{n+1}}{1 - \mu_{n+1}} - \frac{\nu_n}{1 - \mu_n}\right) f(t_n x_n + (1 - t_n)x_{n+1}) \\
&\quad + \frac{\omega_{n+1}}{1 - \mu_{n+1}}[T^{n+1}(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2}) - T^{n+1}(t_n x_n + (1 - t_n)x_{n+1})] \\
&\quad + \left(\frac{\omega_{n+1}}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_n}\right) T^{n+1}(t_n x_n + (1 - t_n)x_{n+1}) \\
&\quad + \frac{\omega_n}{1 - \mu_n}[T^{n+1}(t_n x_n + (1 - t_n)x_{n+1}) - T^n(t_n x_n + (1 - t_n)x_{n+1})].
\end{aligned}$$

因此

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\nu_{n+1}}{1 - \mu_{n+1}} \|f(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2}) - f(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \left|\frac{\nu_{n+1}}{1 - \mu_{n+1}} - \frac{\nu_n}{1 - \mu_n}\right| \|f(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \frac{\omega_{n+1}}{1 - \mu_{n+1}} \|T^{n+1}(t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2}) - T^{n+1}(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \left|\frac{\omega_{n+1}}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_n}\right| \|T^{n+1}(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \frac{\omega_n}{1 - \mu_n} \|T^{n+1}(t_n x_n + (1 - t_n)x_{n+1}) - T^n(t_n x_n + (1 - t_n)x_{n+1}\| \\
&\leq \frac{\nu_{n+1}}{1 - \mu_{n+1}} \gamma \|t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2} - (t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \left|\frac{\nu_{n+1}}{1 - \mu_{n+1}} - \frac{\nu_n}{1 - \mu_{n+1}} + \frac{\nu_n}{1 - \mu_{n+1}} - \frac{\nu_n}{1 - \mu_n}\right| \|f(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \frac{\omega_{n+1}}{1 - \mu_{n+1}} k_{n+1} \|t_{n+1}x_{n+1} + (1 - t_{n+1})x_{n+2} - (t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \left|\frac{\omega_{n+1}}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_{n+1}} + \frac{\omega_n}{1 - \mu_{n+1}} - \frac{\omega_n}{1 - \mu_n}\right| \|T^{n+1}(t_n x_n + (1 - t_n)x_{n+1})\| \\
&\quad + \frac{\omega_n}{1 - \mu_n} \|T^{n+1}(t_n x_n + (1 - t_n)x_{n+1}) - T^n(t_n x_n + (1 - t_n)x_{n+1})\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\nu_{n+1}\gamma}{1-\mu_{n+1}}\|t_n(x_{n+1}-x_n)+(1-t_{n+1})(x_{n+2}-x_{n+1})\| \\
&\quad+\left[\frac{|\nu_{n+1}-\nu_n|}{1-\mu_{n+1}}+\frac{\nu_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right]\|f(x)\|+\frac{\omega_{n+1}}{1-\mu_{n+1}}k_{n+1}\|t_n(x_{n+1}-x_n) \\
&\quad+(1-t_{n+1})(x_{n+2}-x_{n+1})\|+\left[\frac{|\omega_{n+1}-\omega_n|}{1-\mu_{n+1}}+\frac{\omega_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right]\|T^{n+1}x\| \\
&\quad+\frac{\omega_n}{1-\mu_n}\|T^{n+1}x-T^nx\| \\
&\le \frac{\nu_{n+1}\gamma t_n}{1-\mu_{n+1}}\|x_{n+1}-x_n\|+\frac{\nu_{n+1}\gamma(1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+2}-x_{n+1}\| \\
&\quad+\left[\frac{|\nu_{n+1}-\nu_n|}{1-\mu_{n+1}}+\frac{\nu_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right]\|f(x)\|+\frac{\omega_{n+1}k_{n+1}t_n}{1-\mu_{n+1}}\|x_{n+1}-x_n\| \\
&\quad+\frac{\omega_{n+1}k_{n+1}(1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+2}-x_{n+1}\|+\frac{\omega_n}{1-\mu_n}\sup_{x\in C}\|T^{n+1}x-T^nx\| \\
&\quad+\left[\frac{|\omega_{n+1}-\omega_n|}{1-\mu_{n+1}}+\frac{\omega_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right]\|T^{n+1}x\| \\
&\le \frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n}{1-\mu_{n+1}}\|x_{n+1}-x_n\|+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+2}-x_{n+1}\| \\
&\quad+\left(\frac{|\nu_{n+1}-\nu_n|}{1-\mu_{n+1}}+\frac{\nu_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right)\|f(x)\|+\frac{\omega_n}{1-\mu_n}\sup_{x\in C}\|T^{n+1}x-T^nx\| \\
&\quad+\left(\frac{|\omega_{n+1}-\omega_n|}{1-\mu_{n+1}}+\frac{\omega_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right)\|T^{n+1}x\| \\
&\le \frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n}{1-\mu_{n+1}}\|x_{n+1}-x_n\| \\
&\quad+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+2}-x_{n+1}\|+M_{n_2},
\end{aligned}
$$

其中

$$
\begin{aligned}
M_{n_2}=&\left(\frac{|\nu_{n+1}-\nu_n|}{1-\mu_{n+1}}+\frac{\nu_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right)\|f(x)\|+\frac{\omega_n}{1-\mu_n}\sup_{x\in C}\|T^{n+1}x-T^nx\| \\
&+\left(\frac{|\omega_{n+1}-\omega_n|}{1-\mu_{n+1}}+\frac{\omega_n|\mu_n-\mu_{n+1}|}{(1-\mu_{n+1})(1-\mu_n)}\right)\|T^{n+1}x\|.
\end{aligned}
$$

由 (3.2) 得到

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n}{1-\mu_{n+1}}\|x_{n+1}-x_n\| \\
&\quad+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}(\|x_{n+1}-x_n\|+M_{n_1})+M_{n_2} \\
&\le \frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})t_n+(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+1}-x_n\| \\
&\quad+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2} \\
&\le \frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(t_n+1-t_{n+1})}{1-\mu_{n+1}}\|x_{n+1}-x_n\| \\
&\quad+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2}
\end{aligned}
$$

$$\begin{aligned}
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma)-\omega_{n+1}(k_{n+1}-1)}{1-\mu_{n+1}}\right]\|x_{n+1}-x_n\| \\
&\quad+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2} \\
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma)-\varepsilon\nu_{n+1}}{1-\mu_{n+1}}\right]\|x_{n+1}-x_n\|+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2} \\
&\leq \left[1-\frac{\nu_{n+1}(1-\gamma-\varepsilon)}{1-\mu_{n+1}}\right]\|x_{n+1}-x_n\|+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2} \\
&\leq \|x_{n+1}-x_n\|+\frac{(\nu_{n+1}\gamma+\omega_{n+1}k_{n+1})(1-t_{n+1})}{1-\mu_{n+1}}M_{n_1}+M_{n_2}.
\end{aligned}$$

由条件 (i), (ii) 得到

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

根据引理 2.1 得到

$$\lim_{n\to\infty}\|z_n-x_n\|=0.$$

注意

$$x_{n+1}-x_n=(1-\mu_n)(z_n-x_n),$$

所以得到

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.3}$$

**第 3 步** 对每个 $n\in\mathbb{N}$, 我们有

$$\begin{aligned}
&\|x_{n+1}-T^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad=\|\mu_nx_n+\nu_nf(t_nx_n+(1-t_n)x_{n+1})+\omega_nT^n(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad\leq\|\mu_nx_n+\nu_nf(t_nx_n+(1-t_n)x_{n+1})-\mu_nT^n(t_nx_n+(1-t_n)x_{n+1})-\nu_nT^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad\leq\mu_n\|x_n-T^n(t_nx_n+(1-t_n)x_{n+1})\|+\nu_n\|f(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad\leq\mu_n\|x_n-x_{n+1}\|+\mu_n\|x_{n+1}-T^n(t_nx_n+(1-t_n)x_{n+1})\|+\nu_n\|f(t_nx_n+(1-t_n)x_{n+1}) \\
&\qquad-T^n(t_nx_n+(1-t_n)x_{n+1})\|.
\end{aligned}$$

所以

$$\begin{aligned}
&(1-\mu_n)\|x_{n+1}-T^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad\leq\mu_n\|x_n-x_{n+1}\|+\nu_n\|f(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})\|.
\end{aligned}$$

从而

$$\begin{aligned}
&\|x_{n+1}-T^n(t_nx_n+(1-t_n)x_{n+1})\| \\
&\quad\leq\frac{\mu_n}{1-\mu_n}\|x_n-x_{n+1}\|+\frac{\nu_n}{1-\mu_n}\|f(t_nx_n+(1-t_n)x_{n+1})-T^n(t_nx_n+(1-t_n)x_{n+1})\|.
\end{aligned}$$

由 (3.3) 和条件 (i), 得到

$$\lim_{n\to\infty}\|x_{n+1}-T^n(t_nx_n+(1-t_n)x_{n+1})\|=0, \tag{3.4}$$

而且

$$\begin{aligned}
&\|x_n - T^n x_n\| \\
&\quad \le \|x_n - x_{n+1} + x_{n+1} - T^n(t_n x_n + (1-t_n)x_{n+1}) + T^n(t_n x_n + (1-t_n)x_{n+1}) - T^n x_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n(t_n x_n + (1-t_n)x_{n+1})\| + \|T^n(t_n x_n + (1-t_n)x_{n+1}) - T^n x_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n(t_n x_n + (1-t_n)x_{n+1})\| + k_n\|(t_n x_n + (1-t_n)x_{n+1}) - x_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - T^n(t_n x_n + (1-t_n)x_{n+1})\| + k_n(1-t_n)\|x_{n+1} - x_n\|.
\end{aligned}$$

由条件 (iii), (3.3) 和 (3.4), 得到

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.5}$$

我们知道 $T$ 是渐近非扩张映射, 所以

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1} + x_{n+1} - T^{n+1}x_{n+1} + T^{n+1}x_{n+1} - T^{n+1}x_n + T^{n+1}x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + k_{n+1}\|x_{n+1} - x_n\| + k_1\|T^n x_n - x_n\|.
\end{aligned}$$

由 (3.3) 和 (3.5), 得到

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.6}$$

**第 4 步** 因为 $X$ 是 2 一致光滑和一致凸的 Banach 空间, 所以 $X$ 是自反的. 由 $\{x_n\}$ 是有界的, 则存在一个子序列 $\{x_{n_i}\}$ 且 $x_{n_i} \rightharpoonup y$, 所以

$$\lim_{i\to\infty} \langle (I-f)x^*, j(x^* - x_{n_i})\rangle = \limsup_{n\to\infty} \langle (I-f)x^*, j(x^* - x_n)\rangle.$$

由第 3 步和引理 2.2 得到 $y \in F(T)$, 则 $x^* \in F(T)$ 满足

$$\langle (I-f)x^*, j(x^* - y)\rangle \le 0, \quad \forall\, y \in F(T),$$

由对偶映射的弱连续性和引理 2.3, 得到

$$\begin{aligned}
\limsup_{n\to\infty} \langle (I-f)x^*, j(x^* - x_n)\rangle &= \lim_{i\to\infty} \langle (I-f)x^*, j(x^* - x_{n_i})\rangle \\
&= \langle (I-f)x^*, j(x^* - y)\rangle \le 0.
\end{aligned}$$

**第 5 步** 观察

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle \mu_n x_n + \nu_n f(t_n x_n + (1-t_n)x_{n+1}) \\
&\quad + \omega_n T^n(t_n x_n + (1-t_n)x_{n+1}) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \mu_n \langle x_n - x^*, j(x_{n+1} - x^*)\rangle + \nu_n \langle f(t_n x_n + (1-t_n)x_{n+1}) - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad + \omega_n \langle T^n(t_n x_n + (1-t_n)x_{n+1}) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \mu_n \langle x_n - x^*, j(x_{n+1} - x^*)\rangle + \nu_n \langle f(t_n x_n + (1-t_n)x_{n+1}) - f(x^*), j(x_{n+1} - x^*)\rangle \\
&\quad + \nu_n \langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle + \omega_n \langle T^n(t_n x_n + (1-t_n)x_{n+1}) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \mu_n \|x_n - x^*\|\,\|x_{n+1} - x^*\| + \nu_n \|f(t_n x_n + (1-t_n)x_{n+1}) - f(x^*)\|\,\|x_{n+1} - x^*\| \\
&\quad + \nu_n \langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle + \omega_n k_n \|t_n x_n + (1-t_n)x_{n+1} - x^*\|\,\|x_{n+1} - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\le \mu_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| + \nu_n\gamma\|t_n x_n + (1-t_n)x_{n+1} - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle + \omega_n k_n\|t_n x_n + (1-t_n)x_{n+1} - x^*\|\,\|x_{n+1} - x^*\| \\
&\le \mu_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| + \nu_n\gamma t_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + \nu_n\gamma(1-t_n)\|x_{n+1} - x^*\|^2 + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad + \omega_n k_n t_n\|x_n - x^*\|\,\|x_{n+1} - x^*\| + \omega_n k_n(1-t_n)\|x_{n+1} - x^*\|^2 \\
&\le [\mu_n + (\nu_n\gamma + \omega_n k_n)t_n]\|x_n - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + (\nu_n\gamma + \omega_n k_n)(1-t_n)\|x_{n+1} - x^*\|^2 + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le [\mu_n + (\nu_n\gamma + \omega_n k_n)t_n]\left(\frac{\|x_n - x^*\|^2}{2} + \frac{\|x_{n+1} - x^*\|^2}{2}\right) \\
&\quad + (\nu_n\gamma + \omega_n k_n)(1-t_n)\|x_{n+1} - x^*\|^2 + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \frac{\mu_n + (\nu_n\gamma + \omega_n k_n)t_n}{2}\|x_n - x^*\|^2 + \frac{\mu_n + (\nu_n\gamma + \omega_n k_n)(2 - t_n)}{2}\|x_{n+1} - x^*\|^2 \\
&\quad + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle,
\end{aligned}
$$

即

$$
\begin{aligned}
&\left(1 - \frac{\mu_n + (\nu_n\gamma + \omega_n k_n)(2 - t_n)}{2}\right)\|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{\mu_n + (\nu_n\gamma + \omega_n k_n)t_n}{2}\|x_n - x^*\|^2 + \nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle.
\end{aligned}
$$

所以

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{\mu_n + (\nu_n\gamma + \omega_n k_n)t_n}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\|x_n - x^*\|^2 + \frac{2\nu_n\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)} \\
&\le \left[1 - \frac{2[1 - \mu_n - (\nu_n\gamma + \omega_n k_n)(1 - t_n)]}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\right]\|x_n - x^*\|^2 \\
&\quad + \frac{2\nu_n}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \left(1 - \frac{2(1 - \mu_n - \nu_n\gamma - \omega_n k_n)}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\right)\|x_n - x^*\|^2 \\
&\quad + \frac{2\nu_n}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \left(1 - \frac{2[\nu_n(1 - \gamma) - \omega_n(k_n - 1)]}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\right)\|x_n - x^*\|^2 \\
&\quad + \frac{2\nu_n}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \left(1 - \frac{2[\nu_n(1 - \gamma) - \omega_n\varepsilon\nu_n]}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\right)\|x_n - x^*\|^2 \\
&\quad + \frac{2\nu_n}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \left(1 - \frac{2\nu_n(1 - \gamma - \varepsilon)}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\right)\|x_n - x^*\|^2 \\
&\quad + \frac{2\nu_n(1 - \gamma - \varepsilon)}{2 - \mu_n - (\nu_n\gamma + \omega_n k_n)(2 - t_n)}\frac{\langle f(x^*) - x^*, j(x_{n+1} - x^*)\rangle}{1 - \gamma - \varepsilon}.
\end{aligned} \tag{3.7}
$$

令

$$\sigma_n = \frac{2\nu_n(1-\gamma-\varepsilon)}{2-\mu_n-(\nu_n\gamma+\omega_n k_n)(2-t_n)}, \quad \theta_n = \frac{\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle}{1-\gamma-\varepsilon}.$$

注意

$$2-\mu_n-(\nu_n\gamma+\omega_n k_n)(2-t_n) < 2.$$

由条件 (i) 和 (iii) 知

$$\sum_{n=0}^{\infty}\sigma_n = \sum_{n=0}^{\infty}\frac{2\nu_n(1-\gamma-\varepsilon)}{2-\mu_n-(\nu_n\gamma+\omega_n k_n)(2-t_n)} \geq \sum_{n=0}^{\infty}\nu_n(1-\gamma) = +\infty \tag{3.8}$$

和

$$\limsup_{n\to\infty}\theta_n = \limsup_{n\to\infty}\frac{\langle f(x^*)-x^*, j(x_{n+1}-x^*)\rangle}{1-\gamma-\varepsilon} \leq 0. \tag{3.9}$$

根据引理 2.4 和 (3.7)–(3.9), 可得到

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

因此 $x_n$ 强收敛于 $x^* \in F(T)$, 这也是变分不等式问题 (1.1) 的解. 证毕.

**定理 3.2** 设 $C$ 是 Hilbert 空间 $H$ 的一个非空有界闭凸子集, $T: C\to C$ 是一个非扩张映射且 $F(T)\neq\emptyset$, $f$ 是 $C$ 上的 $\gamma\in[0,1)$ 压缩映射. 若 $x_0\in C$, 在 $C$ 上的迭代序列 $\{x_n\}$ 如下定义:

$$x_{n+1} = \mu_n x_n + \nu_n f(t_n x_n + (1-t_n)x_{n+1}) + \omega_n T(t_n x_n + (1-t_n)x_{n+1}), \quad n\geq 0, \tag{3.10}$$

其中, 对于所有的 $n\in\mathbb{N}$, $\mu_n+\nu_n+\omega_n = 1$ 且满足以下条件:

(i) $0\leq \mu_n, \nu_n, \omega_n < 1$, $\lim_{n\to\infty}\nu_n = 0$, $\sum_{n=0}^{\infty}\nu_n = \infty$, $k_n - 1 = \varepsilon\nu_n$, $\varepsilon < 1-\gamma$;

(ii) $\lim_{n\to\infty}|\mu_{n+1}-\mu_n| = 0$, $\lim_{n\to\infty}|\nu_{n+1}-\nu_n| = 0$, $\lim_{n\to\infty}|\omega_{n+1}-\omega_n| = 0$;

(iii) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_{n+1} < 1$, $(\nu_n\gamma+\omega_n k_n)(1-t_n) < 1$, $\forall\, n > 0$,

则 $\{x_n\}$ 强收敛于不动点 $x^*\in F(T)$, 这也是变分不等式问题 (1.1) 的解.

**证明** 我们知道 Hilbert 空间是一致光滑的 Banach 空间, 只需在定理 3.1 中把 Banach 空间换成 Hilbert 空间, 并令 $k_n = 1$ 即可得到结论. 证毕.

**注 3.3** 我们在 Banach 空间和 Hilbert 空间分别提出了广义粘性隐式双中点法则 (3.1) 和 (3.10);

**注 3.4** 我们主要研究了 Banach 空间中一种新的求公共元方法的收敛性分析, 在适当的参数条件下, 得到了一些强收敛定理.

**注 3.5** 在定理 3.1 中令 $\mu_n = 0$, $k_n = 1$, $t_n = \frac{1}{2}$, 这便是 Dhakal [5] 的相关结果.

**注 3.6** 在定理 3.1 中, 若 $k_n \equiv 1$, 则 $T$ 是非扩张映射, 因此, 我们将一些作者的结果从 Hilbert 空间推广到 Banach 空间, 非扩张映射推广到渐近非扩张映射, 粘性隐式双中点法则推广到更广义的粘性隐式双中点法则, 详见文献 [5, 11, 17, 18].

## 参 考 文 献

[1] Bader G., Deuflhard P., A semi-implicit mid-point rule for stiff systems of ordinary differential equations, *Numer. Math.*, 1983, **41**: 373–398.

[2] Cai G., Viscosity iterative methods for new variational inequality problems and fixed point problems in Hilbert spaces, *Acta Math. Sin. Chin. Ser.*, 2019, **62**(5): 765–776.

[3] Cai G., Bu S. Q., Strong convergence theorems of a modified Mann iterative method for non-expansive mapping in Banach spaces (in Chinese), *Acta Math. Sci. Ser. A*, 2014, **34**(1): 1–8.

[4] Cai G., Shehu Y., Iyiola O. S., Modified viscosity implicit rules for non-expansive mappings in Hilbert spaces, *J. Fixed Point Theory Appl.*, 2017, **19**(4): 2831–2846.

[5] Dhakal S., Sintunavarat W., The viscosity method for the implicit double midpoint rule with numerical results and its applications, *Comput. Appl. Math.*, 2019, **38**(2): 1–18.

[6] Iiduka H., Takahashi W., Toyoda M., Approximation of solutions of variational inequalities for monotone mappings, *Pan. Math. J.*, 2004, **14**: 49–61.

[7] Ke Y. F., Ma C. F., The generalized viscosity implicit rules of non-expansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, 2015, **190**: 1–21.

[8] Luo P., Cai G., Shehu Y., The viscosity iterative algorithms for the implicit midpoint rule of non-expansive mappings in uniformly smooth Banach spaces, *J. Inequal. Appl.*, 2017, **154**: 1–12.

[9] Liu C., Liu L. S., Strong convergence of modified Ishikawa iterative algorithm for non-expansive mappings with errors (in Chinese), *Acta Math. Sin. Chin. Ser.*, 2016, **59**(4): 545–560.

[10] Lim T. C., Xu H. K., Fixed point theorems for asymptotically non-expansive mappings, *Nonlinear Anal.*, 1994, **22**: 1345–1355.

[11] Pan C. J., Wang Y. H., Generalized viscosity implicit iterative process for asymptotically non-expansive mappings in Banach spaces, *Mathematics*, 2019, **7**(5): 1–13.

[12] Pan C. J., Wang Y. H., Viscosity approximation methods for a general variational inequality system and fixed point problems in Banach spaces, *Symmetry Basel*, 2020, **12**(1): 1–15.

[13] Somalia S., Implicit midpoint rule to the nonlinear degenerate boundary value problems, *Int. J. Comput. Math.*, 2002, **79**: 327–332.

[14] Song Y., Chen R., Zho H., Viscosity approximation methods for non-expansive mapping sequences in Banach spaces, *Nonlinear Anal.*, 2007, **66**: 1016–1024.

[15] Sunthrayuth P., Kumam P., Viscosity approximation methods base on generalized contraction mappings for a countable family of strict pseudo-contractions, a general system of variational inequalities and a generalized mixed equilibrium problem in Banach spaces, *Math. Comput. Model*, 2013, **58**: 1814–1828.

[16] Wu X., Zhao L., Viscosity approximation methods for multivalued non-expansive mappings, *Mediterr. J. Math.*, 2016, **13**: 2645–2657.

[17] Xu H. K., Alghamdi M., Shahzad N., The viscosity technique for the implicit midpoint rule of non-expansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, 2015, **41**: 1–12.

[18] Yao R. H., Chen R. D., Zhou H. Y., Iterative algorithms for fixed points of non-expansive mappings, *Acta Math. Sin. Chin. Ser.*, 2007, **50**(1): 139–144.

[19] Yao Y. H., Shahzad N., Liou Y. C., Modified semi-implicit midpoint rule for non-expansive mappings, *Fixed Point Theory Appl.*, 2015, **166**: 1–15.

[20] Zhang H. C., Qu Y. H., Su Y. F., The generalized viscosity implicit midpoint rule for non-expansive Mappings in Banach Space, *Mathematics*, 2019, No.512, 16pp.

[21] Zeng L. C., Weak convergence theorems for non-expansive mapping in uniformly convex Banach spaces (in Chinese), *Acta Math. Sin. Sci. Ser. A*, 2002, **22**(3): 336–341.