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# 圆环的加正规权 Bergman 空间上正符号 Toeplitz 算子

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**摘　要**　本文讨论了圆环 $M$ 上正规权 Bergman 空间 $A^p_{\omega_{1,2}}$ $(1<p<\infty)$ 的对偶及其上正符号 Toeplitz 算子, 刻画了这类 Bergman 空间的对偶空间, 并得到了这些正规权 Bergman 空间之间正符号 Toeplitz 算子有界与紧的充要条件.

**关键词**　圆环; 正规权; Bergman 空间; Toeplitz 算子

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## Positive Toeplitz Operators on Bergman Space of Annular Induced by Regular-weight

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**Abstract**　This paper is devoted to studing Bergman spaces induced by regular-weight $A^p_{\omega_{1,2}}(M)$ $(1<p<\infty)$ on annular and positive Toeplitz operators on these spaces. The dual spaces of Bergman spaces induced by regular-weight are characterized. We also

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obtain equivalent conditions for boundedness and compactness of positive Toeplitz operators between these regular-weighted Bergman spaces.

**Keywords** annular; regular-weight; Bergman spaces; Toeplitz operators
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## 1 引言

以下用 $\mathbb{D}$ 表示复平面 $\mathbb{C}$ 中单位圆盘, 则 $M=\mathbb{D}-r_o\overline{\mathbb{D}}$ 为圆环, 为证明方便, 令 $M_1=\{z\,|\,\frac{1+r_0}{2}<|z|<1\}$, $M_2=\{z\,|\,r_0<|z|\leq\frac{1+r_0}{2}\}$, 其中 $0<r_0<1$. $\mathscr{H}(M)$ 为 $M$ 上解析函数全体所成集合. 若 $0<p<\infty$, $\omega_1(z)$, $\omega_2(z)$ 是 $\mathbb{D}$ 上非负可积函数, 令

$$\omega_{1,2}(z)=\omega_1(z)\chi_{\{z\in\mathbb{D}\,|\,\frac{1+r_0}{2}<|z|<1\}}+\omega_2\left(\frac{r_0}{z}\right)\chi_{\{z\in\mathbb{D}\,|\,r_0<|z|\leq\frac{1+r_0}{2}\}},\quad z\in M.$$

定义 $L^p_{\omega_{1,2}}(M)$ 是满足条件

$$\|f\|_p^p=\int_M|f(z)|^p\omega_{1,2}(z)dA(z)<\infty \tag{1.1}$$

的可测函数 $f$ 组成的空间, 其中 $dA(z)=\frac{dxdy}{\pi}$ 是正规化 Lebesgue 面积测度.

易知当 $1\leq p<\infty$ 时, $L^p_{\omega_{1,2}}(M)$ 是 Banach 空间, 而 $L^2_{\omega_{1,2}}(M)$ 为 Hilbert 空间, 其内积为

$$\langle f,g\rangle=\int_M f(z)\overline{g(z)}\omega_{1,2}(z)dA(z).$$

当 $0<p<1$ 时, $L^p_{\omega_{1,2}}(M)$ 是 Frechet 空间. 加权 Bergman 空间定义为 $A^p_{\omega_{1,2}}(M)=L^p_{\omega_{1,2}}(M)\cap\mathscr{H}(M)$.

为了方便, 用 $f\lesssim g$ 表示存在与变量无关的常数 $C>0$, 使得 $f\leq Cg$, 而 $f\asymp g$ 表示 $f\lesssim g$ 与 $g\lesssim f$ 同时成立.

令 $\widehat{\omega}(t)=\int_t^1\omega(s)ds$, $0\leq t<1$. 若径向权函数 $\omega$ 满足倍数条件 $\widehat{\omega}(t)\leq C\widehat{\omega}(\frac{t+1}{2})$, 其中 $C>0$ 为常数, 则记为 $\omega\in\widehat{\mathscr{D}}$. 进而若 $\omega\in\widehat{\mathscr{D}}$ 满足条件

$$\omega(t)\asymp\frac{\int_t^1\omega(s)ds}{1-t},\quad 0\leq t<1,$$

则称 $\omega$ 为正规权函数, 表示为 $\omega\in\mathscr{R}$. 文 [1, (4.4)–(4.6) 式] 给出的权函数均为正规权.

若 $\omega_1$, $\omega_2$ 均为 $\mathbb{D}$ 中径向函数, 则易知对加径向权 Bergman 空间 $A^2_{\omega_{1,2}}(M)$, 按范数收敛可以得到在 $M$ 的紧子集上一致收敛, 于是 $A^2_{\omega_{1,2}}(M)$ 是 $L^2_{\omega_{1,2}}(M)$ 的闭子空间. 从 $L^2_{\omega_{1,2}}(M)$ 到 $A^2_\omega(M)$ 的正交投影 $P_{\omega_{1,2}}$ 是一个积分算子, 形如

$$P_{\omega_{1,2}}f(z)=\int_M f(w)B_z^{\omega_{1,2}}(w)\omega_{1,2}(w)dA(w),$$

其中 $B_z^{\omega_{1,2}}(w)$ 称为 $A^2_{\omega_{1,2}}(M)$ 的再生核.

本文将讨论由投影 $P_{\omega_{1,2}}$ 诱导的积分算子 —Toeplitz 算子. 若 $\mu$ 为 $M$ 上非负 Borel 测度, 以 $\mu$ 为符号的 Toeplitz 算子定义为

$$T_\mu f(z)=\int_M f(w)B_z^{\omega_{1,2}}(w)d\mu(w).$$

若 $d\mu(w)=\phi(w)dA$, 其中 $\phi$ 为非负函数, 则 $T_\mu=T_\phi$, 即 $T_\phi f=P_{\omega_{1,2}}(\phi f)$. 从上世纪 70 年代开始, Hardy 空间、Bergman 空间、Dirichlet 空间等各种解析函数空间上 Toeplitz 算子已经被大量数学家广泛研究, 形成了比较系统的理论, 详见文献 [3, 5, 12]. Luecking 可能最早研究测度符号 Toeplitz 算子, 文 [7] 中刻画了 Toeplitz 算子的 Schatten–p 类性质. 多连通区域的拓扑结构与单连通区域有较大不同, 其上 Bergman 空间上结构也会不同, 于是其上 Toeplitz 算子性质会随之变化, 所以一直受到人们的关注, 且已有了一些有趣的结果 [2, 4, 6].

## 2 Bergman 空间的对偶理论及再生核函数的范数估计和点估计

本节给出文中要用到的一些权函数及其基本性质, 以及再生核函数的范数估计和点估计.

下面介绍一些后面会使用的记号. 若 $I$ 是 $\mathbb{T}=\partial\mathbb{D}$ 上的一段弧, 定义 Carleson 方块 $S(I)$ 为

$$S(I)=\{re^{i\theta}\in\mathbb{D}\,|\,e^{i\theta}\in I,\,1-m(I)\leq r<1\},$$

其中 $m(I)$ 是 $I$ 的正规化 Lebesgue 测度. 当 $a\in\mathbb{D}-\{0\}$, 定义弧 $I_a=\{e^{i\theta}\in\mathbb{T}\,|\,|\arg(ae^{i\theta})|\leq\frac{1-|a|}{2}\}$, 简记 $S(I_a)$ 为 $S(a)$. 因为 $\omega dA$ 是 $\mathbb{D}$ 上的非负 Borel 测度, 故用 $\omega(S(I))$ 表示 $S(I)$ 的测度值, 则 $\omega(S(I))=\int_{S(I)}\omega dA$.

众所周知, $\rho(a,z)=|\phi_a(z)|=|\frac{z-a}{1-\bar{a}z}|$ 为单位圆盘内两点 $a,z$ 之间的拟双曲度量,

$$\Delta(a,r)=\{z\in\mathbb{D}\,|\,\rho(a,z)<r\}$$

表示度量 $\rho$ 的以 $a$ 为心 $r$ 为半径的圆盘, 其中 $a\in\mathbb{D},\ r\in(0,1)$.

满足倍数条件 $\widehat{\mathscr{D}}$ 的权函数性质将会在后面证明中多次使用, 下面的引理来自参考文献 [8]. 首先设 $\widehat{\omega}(t)>0$, 其中 $0\leq t<1$, 否则 $A^p_\omega(\mathbb{D})$ 会退化为 $\mathscr{H}(\mathbb{D})$.

**引理 2.1**　设 $\omega$ 为径向权函数, 则下面条件等价:

(1) $\omega\in\widehat{\mathscr{D}}$;

(2) 存在常数 $C=C(\omega)>0$ 与 $\beta=\beta(\omega)>0$, 使得

$$\widehat{\omega}(t)\leq C\left(\frac{1-t}{1-r}\right)^{\beta}\widehat{\omega}(r),\quad 0\leq t\leq r<1;$$

(3) 存在常数 $C=C(\omega)>0$ 与 $\gamma=\gamma(\omega)>0$, 使得

$$\int_0^t\left(\frac{1-t}{1-s}\right)^{\gamma}\omega(s)ds\leq C\widehat{\omega}(t),\quad 0\leq t<1;$$

(4) 近似等式 $\int_0^1 s^x\omega(s)ds\asymp\widehat{\omega}(1-\frac{1}{x})$, $x\in[1,\infty)$ 成立;

(5) 记 $\omega^*(z)=\int_{|z|}^1\omega(s)\log\frac{s}{|z|}sds$, $z\in\mathbb{D}-\{0\}$, 则 $\omega^*(z)\asymp\widehat{\omega}(z)(1-|z|)$, $|z|\to1^-$;

(6) 存在常数 $\lambda=\lambda(\omega)\geq0$, 使得

$$\int_{\mathbb{D}}\frac{\omega(z)}{|1-z\overline{w}|^{\lambda+1}}\asymp\frac{\widehat{\omega}(w)}{|1-|w|\,|^{\lambda}},\quad w\in\mathbb{D};$$

(7) 存在常数 $C=C(\omega)>0$, 使得 $\omega_n=\int_0^1 r^n\omega(r)dr$ 满足 $\omega_n\leq C\omega_{2n}$.

正规权函数类满足局部光滑性, 且易知若 $\omega\in\mathscr{R}$, 则对任意 $s\in[0,1)$, 存在常数 $C(\omega,s)>1$, 使得

$$C^{-1}\omega(t)\leq\omega(r)\leq C\omega(t),\quad 0\leq r\leq t\leq r+s(1-r)<1.$$

因此

$$\omega(S(z)) \asymp \widehat{\omega}(z)(1-|z|) \asymp \omega(z)(1-|z|)^2 \asymp \omega(\Delta(z)), \quad z \in \mathbb{D},$$

其中最后一个 $\asymp$ 对应的常数与 $\omega, r$ 均有关.

记 $A^2_\omega(\mathbb{D})$ 的再生核为 $K^\omega_z(w)$, 则有如下结论 [11].

**定理 2.2** 设 $\omega, \nu \in \widehat{\mathscr{D}}$, $0<p<\infty$, $n \in \mathbb{N}^*$, 则

$$\|(K^\omega_z(w))^{(n)}\|^p_{A^p_\nu(\mathbb{D})} \asymp \int_0^{|z|} \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p(1-t)^{p(n+1)}}dt, \quad |z| \to 1^-.$$

特别地, 若 $1<p<\infty$, $\omega \in \mathscr{R}$ 且 $r \in (0,1)$, 则

$$\|K^\omega_z(w)\|^p_{A^p_\omega(\mathbb{D})} \asymp \frac{1}{\omega(S(z))^{p-1}} \asymp \frac{1}{\omega(\Delta(z,r))^{p-1}}.$$

后面研究的问题中, 我们均假设 $\omega_1, \omega_2 \in \mathscr{R}$. 为了研究圆环 $M$ 上的 Toeplitz 算子, 需要估计再生核的 $p$ 范数, 这需要对 $A^2_{\omega_{1,2}}(M)$ 的再生核 $B^{\omega_{1,2}}_z(w)$ 做一个拆分, 于是有下面的引理.

**引理 2.3** 对 $r_0 < r_2 < \frac{1+r_0}{2} < r_1 < 1$, 记 $U_2 = \{z \in \mathbb{C} \,|\, r_0 < |z| < r_2\}$ 与 $U_1 = \{z \in \mathbb{C} \,|\, r_1 < |z| < 1\}$. 则当 $(z,w) \in (M \times U_i) \cup (U_i \times M)$ 时, $|B^{\omega_{1,2}}_w(z)| \lesssim |K^{\omega_i}_w(z)| + C(r_i)$, $i=1,2$.

**证明** 因 $B^{\omega_{1,2}}_w(z)$ 在 $M$ 关于 $z$ 解析, 关于 $w$ 共轭解析, 所以

$$B^{\omega_{1,2}}_w(z) = \sum_{k=0}^\infty A_k (z\overline{w})^k + \sum_{j=1}^\infty B_j \left(\frac{r_0^2}{z\overline{w}}\right)^j, \quad (z,w) \in M \times M,$$

其中

$$A_k = \int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt + \int_{r_0}^{\frac{1+r_0}{2}} t^{2k+1}\omega_2\left(\frac{r_0}{t}\right)dt,$$

$$B_j = r_0\int_{\frac{1+r_0}{2}}^1 \left(\frac{r_0}{t}\right)^{2j-1}\omega_1(t)dt + r_0\int_{r_0}^{\frac{1+r_0}{2}} \left(\frac{r_0}{t}\right)^{2j-1}\omega_2\left(\frac{r_0}{t}\right)dt.$$

同理有

$$K^{\omega_1}_w(z) = \sum_{k=0}^\infty c^1_k (z\overline{w})^k, \quad (z,w) \in \mathbb{D} \times \mathbb{D}, \quad K^{\omega_2}_w(z) = \sum_{j=0}^\infty c^2_j (z\overline{w})^j, \quad (z,w) \in \mathbb{D} \times \mathbb{D},$$

其中

$$c^1_k = \int_0^1 t^{2k+1}\omega_1(t)dt, \quad c^2_j = \int_0^1 t^{2j+1}\omega_2(t)dt.$$

因

$$\int_{r_0}^{\frac{1+r_0}{2}} t^{2k+1}\omega_2\left(\frac{r_0}{t}\right)dt \le \left(\frac{1+r_0}{2}\right)^{2k+1}\int_{\frac{2r_0}{1+r_0}}^1 \omega_2(s)\frac{ds}{s^2} \le \left(\frac{1+r_0}{2}\right)^{2k+1}\left(\frac{2r_0}{1+r_0}\right)^{-2}\widehat{\omega_2}\left(\frac{2r_0}{1+r_0}\right),$$

且

$$\int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt \ge \left(\frac{1+r_0}{2}\right)^{2k+1}\widehat{\omega_1}\left(\frac{1+r_0}{2}\right),$$

故

$$c^1_k = \int_0^1 t^{2k+1}\omega_1(t)dt \ge \int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt \gtrsim \int_{r_0}^{\frac{1+r_0}{2}} t^{2k+1}\omega_2\left(\frac{r_0}{t}\right)dt.$$

于是

$$c_k^1=\int_0^1 t^{2k+1}\omega_1(t)dt\gtrsim\int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt+\int_{r_0}^{\frac{1+r_0}{2}} t^{2k+1}\omega_2\left(\frac{r_0}{t}\right)dt=A_k.$$

又因

$$\int_0^{\frac{1+r_0}{2}} t^{2k+1}\omega_1(t)dt\le\left(\frac{1+r_0}{2}\right)^{2k+1}\widehat{\omega_1}(0)\lesssim\left(\frac{1+r_0}{2}\right)^{2k+1}\widehat{\omega_1}\left(\frac{1+r_0}{2}\right)\le\int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt,$$

所以

$$c_k^1=\int_0^1 t^{2k+1}\omega_1(t)dt\lesssim\int_{\frac{1+r_0}{2}}^1 t^{2k+1}\omega_1(t)dt+\int_{r_0}^{\frac{1+r_0}{2}} t^{2k+1}\omega_2\left(\frac{r_0}{t}\right)dt=A_k.$$

由于 $\sum_{j=1}^\infty B_j(\frac{r_0^2}{z\overline{w}})^j$ 在 $\{z\in\mathbb{C}\,|\,|z|>r_0\}\times\{z\in\mathbb{C}\,|\,|z|>r_0\}$ 上关于 $z$ 解析, $w$ 共轭解析, 所以当 $(z,w)\in(M\times U_1)\cup(U_1\times M)$ 时,

$$\left|\sum_{j=1}^\infty B_j\left(\frac{r_0^2}{z\overline{w}}\right)^j\right|\le C(r_1).$$

于是结论成立.

同理有, 当 $(z,w)\in(M\times U_2)\cup(U_2\times M)$ 时, $|B_w^{\omega_{1,2}}(z)|\lesssim|K_w^{\omega_2}(z)|+C(r_2)$. 证毕.

**定理 2.4** 若 $1<p<\infty$, 则 $P_{\omega_{1,2}}$ 是从 $L^p_{\omega_{1,2}}(M)$ 到 $A^p_{\omega_{1,2}}(M)$ 的有界算子.

**证明** 首先证明对任意 $f\in L^p_{\omega_{1,2}}(M)$, $P_{\omega_{1,2}}(f)(z)\in\mathscr{H}(M)$. 注意到, 对任意固定的 $w\in M$, $F(z,w)=B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)$ 是 $M$ 上关于 $z$ 的解析函数, 且因 $M$ 上连续函数在 $L^p_{\omega_{1,2}}(M)$ 中稠密. 同时, 由文 [11] 知, 不改变空间 $L^p_{\omega_{1,2}}(M)$, 适当调整 $\omega_1,\omega_2$ 为连续函数, 于是可得对任意固定 $z\in M$, $F(z,w)$ 是 $M$ 上关于 $w$ 的连续函数. 于是只需证明对任意 $z_0\in M$, $F(z_0,w)\in L^1(M,dA)$.

注意到 $B^{\omega_{1,2}}(z_0,w)\in L^\infty(M)$, 故

$$\begin{aligned}\int_M|F(z_0,w)|dA(w)&\le C(z_0)\int_M|f(w)|\omega_{1,2}(w)dA(w)\\&\lesssim C(z_0)\left(\int_M|f(w)|^p\omega_{1,2}(w)dA(w)\right)^{\frac{1}{p}}<\infty.\end{aligned}$$

现证明 $P_{\omega_{1,2}}$ 的有界性, 若 $f\in L^p_{\omega_{1,2}}(M)$, 则

$$\begin{aligned}&\int_M\left|\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\&\quad\lesssim\int_{U_1}\left|\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\&\qquad+\int_M\left|\int_{U_1} B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\&\qquad+\int_M\left|\int_{U_2} B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\&\qquad+\int_{U_2}\left|\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\&\qquad+\int_{M-U_2-U_1}\left|\int_{M-U_1-U_2} B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z),\end{aligned}$$

其中 $U_1, U_2$ 来自于引理 2.3. 由引理 2.3 知

$$\begin{aligned}
&\int_{U_1}\left|\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\\
&\quad\lesssim\int_{U_1}\left|\int_M (K^{\omega_1}(z,w)+M(r_1))f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_1(z)dA(z)\\
&\quad\lesssim\int_{U_1}\left|\int_M K^{\omega_1}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_1(z)dA(z)\\
&\qquad+\int_{U_1}\left|\int_M M(r_1)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_1(z)dA(z)\\
&\quad\lesssim\int_{U_1}\left|\int_{\mathbb{D}} K^{\omega_1}(z,w)f(w)\chi_{\{z\in\mathbb{D}|\frac{1+r_0}{2}\le|z|<1\}}\omega_1(w)dA(w)\right|^p\omega_1(z)dA(z)\\
&\qquad+\int_{U_1}\left|\int_{\{z\in\mathbb{D}\,|\,r_0<|z|<\frac{1+r_0}{2}\}} K^{\omega_1}(z,w)f(w)\omega_2(w)dA(w)\right|^p\omega_1(z)dA(z)\\
&\qquad+M(r_1)\widehat{\omega_1}(r_1)\|f\|^p_{L^p_{\omega_{1,2}}(M)}\\
&\quad\lesssim\|f\chi_{\{z\in\mathbb{D}|\ \frac{1+r_0}{2}\le|z|<1\}}\|^p_{L^p_{\omega_1}(\mathbb{D})}+M(r_1)\widehat{\omega_1}(r_1)\|f\|^p_{L^p_{\omega_{1,2}}(M)}\\
&\qquad+\operatorname*{Max}_{(z,w)\in\{r_0<|z|<\frac{1+r_0}{2}\}\times U_1}|K^{\omega_1}(z,w)\,|\,\widehat{\omega_1}(r_1)\|f(w)\chi_{\{z\in\mathbb{D}\,|\,r_0<|z|<\frac{1+r_0}{2}\}}\|^p_{L^p_{\omega_2}(\mathbb{D})}\\
&\quad\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)}.
\end{aligned}$$

同理可知

$$\begin{aligned}
&\int_M\left(\int_{U_1}B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right)^p\omega_{1,2}(z)dA(z)\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)},\\
&\int_M\left(\int_{U_2}B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right)^p\omega_{1,2}(z)dA(z)\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)},\\
&\int_{U_2}\left(\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right)^p\omega_{1,2}(z)dA(z)\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)}.
\end{aligned}$$

因 $B^{\omega_{1,2}}(z,w)$ 在 $(M-U_2-U_1)\times(M-U_2-U_1)$ 上有界, 故有

$$\int_{M-U_2-U_1}\left|\int_{M-U_1-U_2}B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)},$$

所以

$$\int_M\left|\int_M B^{\omega_{1,2}}(z,w)f(w)\omega_{1,2}(w)dA(w)\right|^p\omega_{1,2}(z)dA(z)\lesssim\|f\|^p_{L^p_{\omega_{1,2}}(M)}.$$

故 $P_{\omega_{1,2}}$ 有界. 证毕.

**推论 2.5** 若 $1<p<\infty$, $q$ 是 $p$ 的对偶指标, 则对任意 $f\in L^p_{\omega_{1,2}}(M)$, $g\in L^q_{\omega_{1,2}}(M)$, 有

$$\langle P_{\omega_{1,2}}f,g\rangle=\langle f,P_{\omega_{1,2}}g\rangle.$$

**引理 2.6** 若 $1<p<\infty$, 则 $\bigcap_p A^p_{\omega_{1,2}}(M)$ 在 $A^p_{\omega_{1,2}}(M)$ 中稠密.

**证明** 对任意 $f\in A^p_{\omega_{1,2}}(M)$, 则有展式 $f(z)=\sum_{n=0}^\infty a_nz^n+\sum_{k=1}^\infty b_k\frac{r_0^k}{z^k}$. 对 $0<r<1$, 令函数 $f_r(z)=\sum_{n=0}^\infty a_nr^nz^n+\sum_{k=1}^\infty b_k\frac{r_0^kr^k}{z^k}$. 当 $r\to1$ 时, 易知 $f_r(z)$ 在 $M$ 上内闭一致收敛于

$f(z)$. 又因 $f_r(z)$ 在 $M$ 有界, 故 $f_r(z)\in\bigcap_p A^p_{\omega_{1,2}}(M)$. 注意到

$$\begin{aligned}\|f-f_r\|_p^p&=\bigg(\int_M|f(z)-f_r(z)|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\\&\le\bigg(\int_{r_1\le|z|\le r_2}|f(z)-f_r(z)|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\\&\quad+\bigg(\int_{\{r_2<|z|<1\}\cup\{r_0<|z|<r_1\}}|f(z)|^p\omega_{1,2}(z)dA\bigg)^{\frac{1}{p}}\\&\quad+\bigg(\int_{\{r_2<|z|<1\}\cup\{r_0<|z|<r_1\}}|f_r(z)|^p\omega_{1,2}(z)dA\bigg)^{\frac{1}{p}}.\end{aligned}$$

因 $\sum_{n=0}^\infty a_nz^n-\sum_{n=0}^\infty a_nr^nz^n$ 在 $\mathbb{D}$ 解析, $\sum_{k=1}^\infty b_k\frac{r_0^k}{z^k}-\sum_{k=1}^\infty b_k\frac{r_0^kr^k}{z^k}$ 在 $|z|>r_0$ 解析, 所以

$$\begin{aligned}&\bigg(\int_{r_1\le|z|\le r_2}|f(z)-f_r(z)|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\\&\le\bigg(\int_{r_1\le|z|\le r_2}\bigg|\sum_{n=0}^\infty a_nz^n-\sum_{n=0}^\infty a_nr^nz^n\bigg|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\\&\quad+\bigg(\int_{r_1\le|z|\le r_2}\bigg|\sum_{k=1}^\infty b_k\frac{r_0^k}{z^k}-\sum_{k=1}^\infty b_k\frac{r_0^kr^k}{z^k}\bigg|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\\&=\bigg(\int_{r_1}^{r_2}\omega_{1,2}(s)sds\int_0^{2\pi}\bigg|\sum_{n=0}^\infty a_ns^n\mathrm{e}^{\mathrm{i}nt}-\sum_{n=0}^\infty a_nr^ns^n\mathrm{e}^{\mathrm{i}nt}\bigg|^pdt\bigg)^{\frac{1}{p}}\\&\quad+\bigg(\int_{r_1}^{r_2}\omega_{1,2}(s)sds\int_0^{2\pi}\bigg|\sum_{n=0}^\infty b_k\frac{r_0^k}{s^k\mathrm{e}^{\mathrm{i}kt}}-\sum_{n=0}^\infty b_k\frac{r_0^kr^k}{s^k\mathrm{e}^{\mathrm{i}kt}}\bigg|^pdt\bigg)^{\frac{1}{p}}\\&\le\bigg(\int_{r_1}^{r_2}\omega_{1,2}(s)sds\int_0^{2\pi}\bigg|\sum_{n=0}^\infty a_nr_2^n\mathrm{e}^{\mathrm{i}nt}-\sum_{n=0}^\infty a_nr_2^nr^n\mathrm{e}^{\mathrm{i}nt}\bigg|^pdt\bigg)^{\frac{1}{p}}\\&\quad+\bigg(\int_{r_1}^{r_2}\omega_{1,2}(s)sds\int_0^{2\pi}\bigg|\sum_{n=0}^\infty b_k\frac{r_0^k}{r_1^k\mathrm{e}^{\mathrm{i}kt}}-\sum_{n=0}^\infty b_k\frac{r_0^kr^k}{r_1^k\mathrm{e}^{\mathrm{i}kt}}\bigg|^pdt\bigg)^{\frac{1}{p}}.\end{aligned}$$

因 $r\to1$ 时, $\sum_{n=0}^\infty a_nr_2^n\mathrm{e}^{\mathrm{i}nt}-\sum_{n=0}^\infty a_nr_2^nr^n\mathrm{e}^{\mathrm{i}nt}\to0$ 且 $\sum_{n=0}^\infty b_k\frac{r_0^k}{r_1^k\mathrm{e}^{\mathrm{i}kt}}-\sum_{n=0}^\infty b_k\frac{r_0^kr^k}{r_1^k\mathrm{e}^{\mathrm{i}kt}}\to0$, 故

$$\bigg(\int_{r_1\le|z|\le r_2}|f(z)-f_r(z)|^p\omega_{1,2}(z)dA(z)\bigg)^{\frac{1}{p}}\to0.$$

由一致收敛性知, 当 $r\to1$ 时, $(\int_{r_1\le|z|\le r_2}|f(z)-f_r(z)|\omega_{1,2}(z)dA(z))^{\frac{1}{p}}\to0$. 由 $f,f_r\in A^p_{\omega_{1,2}}(M)$ 与积分绝对连续性, 可取充分接近 $r_0$ 与 1 的 $r_1$ 与 $r_2$, 使得对任意 $\varepsilon>0$, 有

$$\bigg(\int_{\{r_2<|z|<1\}\cup\{r_0<|z|<r_1\}}|f(z)|^p\omega_{1,2}(z)dA\bigg)^{\frac{1}{p}}+\bigg(\int_{\{r_2<|z|<1\}\cup\{r_0<|z|<r_1\}}|f_r(z)|^p\omega_{1,2}(z)dA\bigg)^{\frac{1}{p}}<\varepsilon.$$

于是结论成立. 证毕.

**定理 2.7**　若 $1<p<\infty$, 则对任意 $f\in L^p_{\omega_{1,2}}(M)$, 有

$$P_{\omega_{1,2}}f=f.$$

**证明** 由引理 2.6 知, $\bigcap_p A^p_{\omega_{1,2}}(M)$ 在 $A^p_{\omega_{1,2}}(M)$ 中稠密, 故对任意 $f \in A^p_{\omega_{1,2}}(M)$, 存在 $f_n \in A^2_{\omega_{1,2}}(M)$, 使得当 $n \to \infty$ 时,

$$\|f_n - f\|_p \to 0.$$

又因 $P_{\omega_{1,2}}$ 在 $L^p_{\omega_{1,2}}(M)$ 上有界, 所以当 $n \to \infty$ 时, $\|P_{\omega_{1,2}} f_n - P_{\omega_{1,2}} f\|_p \to 0$. 于是

$$P_{\omega_{1,2}} f(z) = \lim_{n\to\infty} P_{\omega_{1,2}} f_n(z) = \lim_{n\to\infty} f_n(z) = f(z).$$

证毕.

上述结果说明 $P_{\omega_{1,2}}$ 是 $L^p_{\omega_{1,2}}(M)$ 到 $A^p_{\omega_{1,2}}(M)$ 上的有界投影, 于是可得下面的插值定理.

**定理 2.8** 若 $1 < p_0 < p_1 < \infty$, $0 \le \theta \le 1$, 则

$$[L^{p_0}_{\omega_{1,2}}(M), L^{p_1}_{\omega_{1,2}}(M)]_\theta = L^p_{\omega_{1,2}}(M), \quad [A^{p_0}_{\omega_{1,2}}(M), A^{p_1}_{\omega_{1,2}}(M)]_\theta = A^p_{\omega_{1,2}}(M),$$

其中 $p$ 满足 $\frac{1}{p} = \frac{1-\theta}{p} + \frac{\theta}{p}$.

**证明** 由复插值理论知 $[L^{p_0}_{\omega_{1,2}}(M), L^{p_1}_{\omega_{1,2}}(M)]_\theta = L^p_{\omega_{1,2}}(M)$ 显然成立. 因 $P_{\omega_{1,2}}$ 是 $L^p_{\omega_{1,2}}(M)$ 到 $A^p_{\omega_{1,2}}(M)$ 上的有界投影, 故在范数等价的意义下, 有

$$\begin{aligned}
[A^{p_0}_{\omega_{1,2}}(M), A^{p_1}_{\omega_{1,2}}(M)]_\theta &= [P_{\omega_{1,2}} L^{p_0}_{\omega_{1,2}}(M), P_{\omega_{1,2}} L^{p_1}_{\omega_{1,2}}(M)]_\theta \\
&= P_{\omega_{1,2}}([L^{p_0}_{\omega_{1,2}}(M), L^{p_1}_{\omega_{1,2}}(M)]_\theta) = P_{\omega_{1,2}} L^p_{\omega_{1,2}}(M) \\
&= A^p_{\omega_{1,2}}(M).
\end{aligned}$$

当 $1 < p < \infty$ 时, 若 $q > 1$ 满足 $\frac{1}{p} + \frac{1}{q} = 1$, 则 $q$ 称为 $p$ 的共轭指标, $L^p$ 共轭理论说明在配对积分

$$\langle f, g\rangle = \int_M f(z)\overline{g(z)}\omega_{1,2}(z)dA(z)$$

意义下, $L^p_{\omega_{1,2}}(M)$ 的对偶空间为 $L^q_{\omega_{1,2}}(M)$. 证毕.

于是由 $P_{\omega_{1,2}}$ 的有界性可得到下面 $A^p_{\omega_{1,2}}(M)$ 的对偶理论.

**定理 2.9** 若 $1 < p < \infty$, $q$ 为 $p$ 的共轭指标, 则在配对积分 $\langle f, g\rangle = \int_M f(z)\overline{g(z)}\omega_{1,2}(z)dA(z)$ 意义下

$$(A^p_{\omega_{1,2}}(M))^* \cong A^q_{\omega_{1,2}}(M).$$

**证明** 易知任意 $g \in A^q_{\omega_{1,2}}(M)$ 在 $\langle f, g\rangle$ 意义下诱导 $A^p_{\omega_{1,2}}(M)$ 上有界线性泛函 $S_g$, 且满足

$$\|S_g\| \le \|g\|_q.$$

若 $S \in (A^p_{\omega_{1,2}}(M))^*$, 由 Hahn–Banach 定理知 $S$ 可以保范延拓到 $L^p_{\omega_{1,2}}(M)$ 上, 即得到 $L^p_{\omega_{1,2}}(M)$ 上有界线性泛函 $S'$ 满足 $S'|_{A^p_{\omega_{1,2}}(M)} = S$ 且 $\|S'\| = \|S\|$. 因此存在 $G \in L^q_{\omega_{1,2}}(M)$, 使得 $S'_G = S'$. 因对任意 $f \in A^p_{\omega_{1,2}}(M)$, 有 $P_{\omega_{1,2}} f = f$ 且 $P_{\omega_{1,2}}$ 在 $L^p_{\omega_{1,2}}(M)$ 上有界, 则

$$S(f) = S'(f) = S'_G(f) = \langle f, G\rangle = \langle P_{\omega_{1,2}} f, G\rangle = \langle f, P_{\omega_{1,2}} G\rangle,$$

其中 $g = P_{\omega_{1,2}} G \in A^q_{\omega_{1,2}}(M)$ 且 $\|g\|_q \lesssim \|G\|_q = \|S'\| = \|S\|$. 证毕.

为了给出 $A^2_{\omega_{1,2}}(M)$ 上再生核 $B^{\omega_{1,2}}_z(w)$ 的点估计与范数估计, 先介绍下面定理, 此结果来自于文 [9, 定理 1].

**定理 2.10** 若 $\nu,\omega\in\widehat{\mathscr{D}}$, $0<p<\infty$ 且 $n\in\mathbb{N}^*$, 则

$$\|(K_z^{\omega}(w))^{(n)}\|_{A_\nu^p}^p\asymp\int_0^{|z|}\frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p(1-t)^{p(n+1)}}dt,\quad |z|\to1^-.$$

特别地, 若 $1<p<\infty$, $\omega\in\mathscr{R}$ 且 $0<r<1$, 则

$$\|K_z^{\omega}(w)\|_{A_\omega^p}^p\asymp\frac{1}{\omega(S(z))^{p-1}}\asymp\frac{1}{\omega(\Delta(z,r))^{p-1}}.$$

利用上面的定理, 再参考引理 2.3 证明可得到下面关于 $B_z^{\omega_{1,2}}(w)$ 的范数估计.

**定理 2.11** 若 $\omega_1,\omega_2\in\mathscr{R}$, $1<p<\infty$ 且 $0<r<1$, 则

$$\|B_w^{\omega_{1,2}}(z)\|_p^p\asymp\frac{1}{\omega_1(S(w))^{p-1}}\asymp\frac{1}{\omega_1(\Delta(w,r))^{p-1}},\quad |w|\to1^-,$$

且

$$\|B_w^{\omega_{1,2}}(z)\|_p^p\asymp\frac{1}{\omega_2(S(\frac{r_0}{w}))^{p-1}}\asymp\frac{1}{\omega_2(\Delta(\frac{r_0}{w},r))^{p-1}},\quad |w|\to r_0^+.$$

**证明** 若

$$B_w^{\omega_{1,2}}(z)=\sum_{k=0}^{\infty}A_k(z\overline{w})^k+\sum_{j=1}^{\infty}B_j\left(\frac{r_0^2}{z\overline{w}}\right)^j,\quad (z,w)\in M\times M,$$

$$K_w^{\omega_1}(z)=\sum_{k=0}^{\infty}c_k^1(z\overline{w})^k,\quad K_w^{\omega_2}(z)=\sum_{k=0}^{\infty}c_k^2(z\overline{w})^k,$$

则由引理 2.3 的证明知 $A_k\asymp c_k^1$ 且 $B_k\asymp c_k^2$. 于是有

$$|B_w^{\omega_{1,2}}(z)|\asymp\left|K_w^{\omega_1}(z)+K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)-c_0^2\right|,\quad (z,w)\in M\times M,$$

所以

$$\begin{aligned}
\|B_z^{\omega_{1,2}}(w)\|_p^p&=\int_M|B_w^{\omega_{1,2}}(z)|^p\omega_{1,2}(z)dA(z)\\
&=\int_{r_0<|z|\le\frac{1+r_0}{2}}|B_w^{\omega_{1,2}}(z)|^p\omega_2\left(\frac{r_0}{z}\right)dA(z)+\int_{\frac{1+r_0}{2}<|z|<1}|B_w^{\omega_{1,2}}(z)|^p\omega_2(z)dA(z)\\
&\lesssim\int_{r_0<|z|\le\frac{1+r_0}{2}}|K_w^{\omega_1}(z)|^p\omega_2\left(\frac{r_0}{z}\right)dA(z)+\int_{r_0<|z|\le\frac{1+r_0}{2}}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|^p\omega_2\left(\frac{r_0}{z}\right)dA(z)\\
&\quad+\int_{\frac{1+r_0}{2}<|z|<1}|K_w^{\omega_1}(z)|^p\omega_1(z)dA(z)+\int_{\frac{1+r_0}{2}<|z|<1}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|^p\omega_1(z)dA(z)\\
&\lesssim\|K_w^{\omega_1}\|_p^p+\|K_{\frac{r_0}{w}}^{\omega_2}\|_p^p+\operatorname*{Max}_{\frac{1+r_0}{2}<|z|<1,\,w\in M}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|\widehat{\omega_1}(r_0)\\
&\quad+\operatorname*{Max}_{r_0<|z|\le\frac{1+r_0}{2},\,w\in M}|K_w^{\omega_1}(z)|\widehat{\omega_2}\left(\frac{2r_0}{1+r_0}\right).
\end{aligned}$$

事实上, 上式中的

$$\operatorname*{Max}_{\frac{1+r_0}{2}<|z|<1,\,w\in M}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|\widehat{\omega_1}(r_0)+\operatorname*{Max}_{r_0<|z|\le\frac{1+r_0}{2},\,w\in M}|K_w^{\omega_1}(z)|\widehat{\omega_2}\left(\frac{2r_0}{1+r_0}\right)$$

是常数.

另一方面, 由

$$\begin{aligned}\|K_w^{\omega_1}\|_p^p+\|K_{\frac{r_0}{w}}^{\omega_2}\|_p^p\le\|B_z^{\omega_{1,2}}(w)\|_p^p&+\operatorname*{Max}_{\frac{1+r_0}{2}<|z|<1,\,w\in M}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|\widehat{\omega_1}(r_0)\\&+\operatorname*{Max}_{r_0<|z|\le\frac{1+r_0}{2},\,w\in M}|K_w^{\omega_1}(z)|\widehat{\omega_2}\left(\frac{2r_0}{1+r_0}\right)\\&+\int_{|z|\le\frac{r_0+1}{2}}|K_w^{\omega_1}(z)|^p\omega_1(z)dA(z)+\int_{\frac{r_0+1}{2}<|z|<1}\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)\right|^p\omega_2\left(\frac{r_0}{z}\right)dA(z),\end{aligned}$$

易知 $\int_{|z|\le\frac{r_0+1}{2}}|K_w^{\omega_1}(z)|^p\omega_1(z)dA(z)+\int_{\frac{r_0+1}{2}<|z|<1}|K_{\frac{r_0}{w}}^{\omega_2}(\frac{r_0}{z})|^p\omega_2(\frac{r_0}{z})dA(z)$ 为常数.

注意到, 当 $|w|\to1^-$ 时, $\|K_{\frac{r_0}{w}}^{\omega_2}\|_p^p$ 有界, 故

$$\|B_w^{\omega_{1,2}}(z)\|_p^p\asymp\frac{1}{\omega_1(S(w))^{p-1}}\asymp\frac{1}{\omega_1(\Delta(w,r))^{p-1}}.$$

同理, 当 $|w|\to r_0^+$ 时,

$$\|B_w^{\omega_{1,2}}(z)\|_p^p\asymp\frac{1}{\omega_2(S(\frac{r_0}{w}))^{p-1}}\asymp\frac{1}{\omega_2(\Delta(\frac{r_0}{w},r))^{p-1}},\quad|w|\to r_0^+.$$

证毕.

**引理 2.12** 设 $\omega_1,\omega_2\in\widehat{\mathscr{D}}$, 则存在 $r=r(\omega_{1,2})\in(0,1)$, 使得

$$|B_w^{\omega_{1,2}}(z)|\asymp B_z^{\omega_{1,2}}(z),\quad z\in M,\quad w\in M\cap\Delta(z,r)\ \text{且}\ \frac{r_0}{w}\in M\cap\Delta\left(\frac{r_0}{z},r\right).$$

**证明** 由 Cauchy–Schwarz 不等式与文 [11, 引理 8] 知, $z\in M$, $w\in M\cap\Delta(z,r)$ 且 $\frac{r_0}{w}\in M\cap\Delta(\frac{r_0}{z},r)$ 时, 有

$$|B_w^{\omega_{1,2}}(z)|\lesssim|K_w^{\omega_1}(z)|+\left|K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)-c_0^2\right|\asymp K_z^{\omega_1}(z)+K_{\frac{r_0}{z}}^{\omega_2}\left(\frac{r_0}{z}\right)-c_0^2=B_z^{\omega_{1,2}}(z).$$

反过来, 对 $r\in(0,1)$

$$\begin{aligned}|B_w^{\omega_{1,2}}(z)|&\asymp\left|K_w^{\omega_1}(z)+K_{\frac{r_0}{w}}^{\omega_2}\left(\frac{r_0}{z}\right)-c_0^2\right|\\&\ge B_z^{\omega_{1,2}}(z)-\operatorname*{Max}_{\zeta\in[z,w]}|(K_w^{\omega_1})'(\zeta)|\,|z-w|-\operatorname*{Max}_{\zeta\in[\frac{r_0}{z},\frac{r_0}{w}]}\left|(K_{\frac{r_0}{w}}^{\omega_2})'\left(\frac{r_0}{\zeta}\right)\right|\left|\frac{r_0}{z}-\frac{r_0}{w}\right|\\&\ge B_z^{\omega_{1,2}}(z)-\operatorname*{Max}_{\zeta\in[z,w]}|(K_w^{\omega_1})'(\zeta)|rC|\,|z|-1|-\operatorname*{Max}_{\zeta\in[\frac{r_0}{z},\frac{r_0}{w}]}\left|(K_{\frac{r_0}{w}}^{\omega_2})'\left(\frac{r_0}{\zeta}\right)\right|rC\left|\frac{r_0}{|z|}-1\right|,\end{aligned}$$

其中 $C=C(r)>0$为常数, 满足对某个给定常数 $1>a>0$, 当 $0<r<a<1$ 时, $C(r)<M<\infty$. 运用 Cauchy 积分公式与文 [11, 引理 8], 得

$$\operatorname*{Max}_{\zeta\in[z,w]}|(K_w^{\omega_1})'(\zeta)|\lesssim\frac{K_z^{\omega_1}(z)}{1-|z|},\quad\text{且}\quad\operatorname*{Max}_{\zeta\in[\frac{r_0}{z},\frac{r_0}{w}]}\left|(K_{\frac{r_0}{w}}^{\omega_2})'\left(\frac{r_0}{\zeta}\right)\right|\lesssim\frac{K_{\frac{r_0}{z}}^{\omega_2}(\frac{r_0}{z})}{|\frac{r_0}{|z|}-1|}.$$

于是

$$|B_w^{\omega_{1,2}}(z)|\asymp B_z^{\omega_{1,2}}(z),\quad z\in M,\quad w\in M\cap\Delta(z,r),\quad\text{且}\ \frac{r_0}{w}\in M\cap\Delta\left(\frac{r_0}{z},r\right).$$

证毕.

**引理 2.13** 设 $0 < p < \infty$, $\omega_1, \omega_2 \in \widehat{\mathscr{D}}$. 若 $f \in A^p_{\omega_{1,2}}$, 则当 $z \to 1^-$ 时, $|f(z)| = o(\frac{1}{\widehat{\omega_1}(z)(1-|z|)^{\frac{1}{p}}})$; 当 $z \to r_0^+$ 时, $|f(z)| = o(\frac{1}{\widehat{\omega_2}(\frac{r_0}{z})(1-|\frac{r_0}{z}|)^{\frac{1}{p}}})$.

**证明** 因为 $f \in A^p_{\omega_{1,2}}$, 故由洛朗展式

$$f(z) = \sum_{k=0}^{\infty} h_k z^k + \sum_{n=1}^{\infty} h_{-n} \frac{r_0^n}{z^n},$$

易知 $\sum_{k=0}^{\infty} h_k z^k$ 在 $\mathbb{D}$ 内闭一致收敛. 同理 $\sum_{n=1}^{\infty} h_{-n}\frac{r_0^n}{z^n}$ 在 $\{z \in \mathbb{C} \,|\, |z| > r_0\}$ 内闭一致收敛, 于是可令 $f_1(z) = \sum_{k=0}^{\infty} h_k z^k$, $f_2(\frac{r_0}{z}) = \sum_{n=1}^{\infty} h_{-n}\frac{r_0^n}{z^n}$. 注意到

$$\begin{aligned}
\|f_1\|^p_{A^p_{\omega_1}} &= \int_{\frac{1+r_0}{2}<|z|<1} |f_1(z)|^p \omega_1(z) dA(z) + \int_{|z|\le\frac{1+r_0}{2}} |f_1(z)|^p \omega_1(z) dA(z) \\
&\le \int_{|z|\le\frac{1+r_0}{2}} |f_1(z)|^p \omega_1(z) dA(z) + \|f(z)\|^p_{A^p_{\omega_{12}}} + \int_{\frac{1+r_0}{2}<|z|<1} \left|f_2\left(\frac{r_0}{z}\right)\right|^p \omega_1(z) dA(z) \\
&\le M\left(\frac{1+r_0}{2}, |f_1|^p\right) \int_{\frac{1+r_0}{2}}^{1} \omega_1(t) t dt + \|f(z)\|^p_{A^p_{\omega_{12}}} \\
&\quad + M\left(\frac{2r_0}{1+r_0}, |f_2|^p\right) \int_{r_0}^{\frac{2r_0}{1+r_0}} \omega_1\left(\frac{r_0}{t}\right)\frac{r_0^2}{t^3} dt + \|f(z)\|^p_{A^p_{\omega_{12}}},
\end{aligned}$$

因为 $f_1(z)$ 在 $\mathbb{D}$ 内解析, $f_2(\frac{r_0}{z})$ 在 $\{z \in \mathbb{C} \,|\, |z| > r_0\}$ 内解析, 故 $M(\frac{1+r_0}{2}, |f_1|^p)$ 为有限常数, $M(\frac{2r_0}{1+r_0}, |f_2|^p)$ 为有限常数. 所以 $\|f_1\|^p_{A^p_{\omega_1}} < \infty$, 即 $f_1 \in A^p_{\omega_1}$. 同理, $f_2(\frac{r_0}{z}) \in A^p_{\omega_2}$. 由文 [11, 引理 9] 知

$$|f(z)| = o\left(\frac{1}{\widehat{\omega_1}(z)(1-|z|)^{\frac{1}{p}}}\right);$$

当 $z \to r_0^+$ 时,

$$|f(z)| = o\left(\frac{1}{\widehat{\omega_2}(\frac{r_0}{z})(1-|\frac{r_0}{z}|)^{\frac{1}{p}}}\right).$$

证毕.

**引理 2.14** 设 $1 < p < \infty$, $\omega_1, \omega_2 \in \mathscr{R}$, 则当 $|w| \to 1^-$ 或 $|w| \to r_0^+$ 时, 有

$$b^{\omega_{1,2}}_{p,w} = \frac{B^{\omega_{1,2}}_w}{\|B^{\omega_{1,2}}_w\|_p} \xrightarrow{w} 0.$$

**证明** 若 $1 < p < \infty$, $\omega_1, \omega_2 \in \mathscr{R}$, 则对任意 $g \in A^{p'}_{\omega_{1,2}}(M)$ (其中 $p'$ 满足 $\frac{1}{p} + \frac{1}{p'} = 1$), 有

$$|\langle b^{\omega_{1,2}}_{p,w}, g\rangle| = \frac{g(w)}{\|B^{\omega_{1,2}}_w\|_p}.$$

由定理 2.11 知

$$\|B^{\omega_{1,2}}_w(z)\|_p \asymp \frac{1}{\omega_1(S(w))^{\frac{1}{p'}}} \asymp \frac{1}{\omega_1(\Delta(w,r))^{\frac{1}{p'}}}, \quad |w| \to 1^-,$$

且

$$\|B^{\omega_{1,2}}_w(z)\|_p \asymp \frac{1}{\omega_2(S(\frac{r_0}{w}))^{\frac{1}{p'}}} \asymp \frac{1}{\omega_2(\Delta(\frac{r_0}{w}, r))^{\frac{1}{p'}}}, \quad |w| \to r_0^+.$$

结合引理 2.1 (7) 与引理 2.13, 得

$$\frac{g(w)}{\|B^{\omega_{1,2}}_w\|_p} \to 0, \quad |w| \to r_0^+ \text{ 或 } |w| \to 1^-.$$

证毕.

## 3 有界与紧 Toeplitz 算子

本节主要讨论从 Bergman 空间 $A^p_{\omega_{1,2}}(M)$ 到 $A^q_{\omega_{1,2}}(M)$ 的有界与紧 Toeplitz 算子, 利用 Berezin 变换和均值函数给出 Toeplitz 算子是有界与紧的等价刻画.

**引理 3.1** 设 $\mu$ 为 $M$ 上有限正 Borel 测度. 若 $f(z)=\sum_{k=0}^{\infty} f_k z^k+\sum_{n=1}^{\infty} f_{-n}\frac{r_0^n}{z^n}$ 与 $g(z)=\sum_{k=0}^{\infty} g_k z^k+\sum_{n=1}^{\infty} g_{-n}\frac{r_0^n}{z^n}$ 满足 $f\in H^{\infty}(M)$, $g(z)=\sum_{k=0}^{\infty}|g_k|+\sum_{n=1}^{\infty}|g_{-n}|<\infty$, 则

$$\langle T_\mu f, g\rangle=\int_M f(\zeta)\overline{g(\zeta)}d\mu(\zeta).$$

**证明**

$$\begin{aligned}\langle T_\mu f, g\rangle&=\lim_{r_1\to 1,\, r_2\to r_0}\int_{r_2<|z|<r_1}\left(\int_M f(w)B_z^{\omega_{1,2}}(w)d\mu(w)\right)\overline{g(z)}\omega_{1,2}(z)dA(z)\\&=\lim_{r_1\to 1,\, r_2\to r_0}\int_M f(w)\left(\int_{r_2<|z|<r_1}\overline{g(z)}B_z^{\omega_{1,2}}(w)\omega_{1,2}(z)dA(z)\right)d\mu(w)\\&=\int_M f(w)\overline{g(w)}d\mu(w).\end{aligned}$$

记 $b_w^{\omega_{1,2}}=b_{2,w}^{\omega_{1,2}}=\frac{B_w^{\omega_{1,2}}}{\|B_w^{\omega_{1,2}}\|_2}$, 定义有限正 Borel 测度 $\mu$ 的 Berezin 变换为

$$\widetilde{\mu}(w)=\widetilde{T_\mu}(w)=\langle T_\mu b_w^{\omega_{1,2}}, b_w^{\omega_{1,2}}\rangle=\frac{\|B_w^{\omega_{1,2}}\|^2_{L^2(M,d\mu)}}{\|B_w^{\omega_{1,2}}\|_2^2}.$$

由定理 2.11 知: 当 $|w|\to 1^-$ 时,

$$\widetilde{\mu}(w)\asymp\|B_w^{\omega_{1,2}}\|^2_{L^2(M,d\mu)}\omega_1(S(w));$$

当 $|w|\to r_0^+$ 时,

$$\widetilde{\mu}(w)\asymp\|B_w^{\omega_{1,2}}\|^2_{L^2(M,d\mu)}\omega_1\left(S\left(\frac{r_0}{w}\right)\right).$$

证毕.

当 $1\le p,\, q<\infty$ 时, 对任意 $f\in A^p_{\omega_{1,2}}(M)$, 非负 Borel 测度 $\mu$ 满足不等式

$$\left(\int_M |f(z)|^q d\mu(z)\right)^{\frac{1}{q}}\le\|f\|_p,$$

则称 $\mu$ 为 $(p,q)$-Carleson 测度, 这意味着恒等映射

$$I_\mu: A^p_{\omega_{1,2}}\to L^q(M,d\mu)$$

是有界映射.

若 $\{f_n\}\subset A^p_{\omega_{1,2}}(M)$ 为有界序列, 且在 $M$ 的任意紧子集上当 $n\to\infty$ 时, $f_n$ 一致收敛于 0, 有

$$\int_M |f_n(z)|^q d\mu(z)\to 0,$$

则称 $\mu$ 为 $(p,q)$- 消失 Carleson 测度, 这意味着恒等映射

$$I_\mu: A^p_{\omega_{1,2}}\to L^q(M,d\mu)$$

为紧.

利用解析函数模的次调和性质, 可以得到下面的点估计.

**引理 3.2** 若函数 $f$ 在 $M$ 上解析, $\omega\in\mathscr{R}$ 且 $0<p<\infty$, $r>0$, 则

$$|f(z)|\lesssim\left(\frac{1}{\omega_1(\Delta(z,r))}\int_{\Delta(z,r)}|f(w)|^p\omega_1(w)dA(w)\right)^{\frac{1}{p}},\quad z\in M_1,$$

且

$$|f(z)|\lesssim\left(\frac{1}{\omega_1(\Delta(\frac{r_0}{z},r))}\int_{\Delta(\frac{r_0}{z},r)}|f(w)|^p\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)^{\frac{1}{p}},\quad z\in M_2.$$

**证明** 当 $z\in M_1$, 利用 $|f(z)|^p$ 的次调和性质, 可得

$$|f(z)|\lesssim\left(\frac{1}{|\Delta(z,r)|}\int_{\Delta(z,r)}|f(w)|^pdA(w)\right)^{\frac{1}{p}}\asymp\left(\frac{1}{(1-|z|)^2}\int_{\Delta(z,r)}|f(w)|^pdA(w)\right)^{\frac{1}{p}}.$$

由文 [11, (2.2)] 得 $\omega_1(\Delta(z,r))\asymp\omega_1(z)(1-|z|)^2$, 又因当 $w\in\Delta(z,r)$ 时, $\omega_1(z)\asymp\omega_1(w)$, 于是有

$$|f(z)|\lesssim\left(\frac{1}{\omega_1(\Delta(z,r))}\int_{\Delta(z,r)}|f(w)|^p\omega_1(w)dA(w)\right)^{\frac{1}{p}}.$$

同理可得另一个不等式. 证毕.

下面将研究由有限正 Borel 测度诱导的 Toeplitz 算子的有界性.

**定理 3.3** 若 $1<p\leq q<\infty$ 且 $\mu$ 为 $M$ 上有限正 Borel 测度, 则下面条件等价:

(1) $T_\mu: A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 有界;

(2) $\frac{\widetilde{T_\mu}(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}\in L^\infty(M_1)$, $\frac{\widetilde{T_\mu}(w)}{\omega_2(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}}\in L^\infty(M_2)$, 其中 $q'$ 是 $q$ 的共轭数;

(3) 对 $0<s<\infty$ 中某个 (全部的) $s$, $\mu$ 是空间 $A^s_{\omega_{1,2}}$ 的 $\frac{s(p+q')}{pq'}$-Carleson 测度;

(4) $\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\in L^\infty(M_1)$, $\frac{\mu(S(\frac{r_0}{w}))}{\omega_2(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\in L^\infty(M_2)$.

进一步, 有

$$\begin{aligned}\|T_\mu\|&\asymp\left\|\frac{\widetilde{T_\mu}(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}\right\|_{L^\infty(M_1)}+\left\|\frac{\widetilde{T_\mu}(w)}{\omega_2(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}}\right\|_{L^\infty(M_2)}\\&\asymp\|I\|^{s\frac{p+q'}{pq'}}_{A^s_{\omega_{1,2}}\to L^{\frac{s(p+q')}{pq'}}(M,d\mu)}\asymp\left\|\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\right\|_{L^\infty(M_1)}+\left\|\frac{\mu(S(\frac{r_0}{w}))}{\omega_1(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\right\|_{L^\infty(M_2)}.\end{aligned}$$

**证明** 首先证明 (3) 可以推出 (4). 当 $s>0$, 对任意 $f\in A^s_{\omega_1}(\mathbb{D})$, 令 $M_2=\{z\,|\,r_0<|z|\leq\frac{1+r_0}{2}\}$, $M_1=\{z\,|\,\frac{1+r_0}{2}<|z|<1\}$, $M_p(f,r)=\int_0^{2\pi}|f(re^{\mathrm{i}t})|^pdt$, 则显然 $f\in A^s_{\omega_{1,2}}(M)$. 因 $\mu$ 可以延拓为 $\mathbb{D}$ 上测度, 使得支集为 $M$, 所以

$$\begin{aligned}\left(\int_{\mathbb{D}}|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}&\leq\left(\int_M|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\lesssim\left(\int_M|f|^s\omega_{1,2}dA\right)^{\frac{1}{s}}\\&\lesssim\left(\int_{M_1}|f|^s\omega_1dA\right)^{\frac{1}{s}}+\left(M_s(f,\frac{1+r_0}{2})\widehat{\omega_2}(r_0)\right)^{\frac{1}{s}}\\&\lesssim\left(\int_{M_1}|f|^s\omega_1dA\right)^{\frac{1}{s}}+\left(\frac{\widehat{\omega_2}(r_0)}{\widehat{\omega_1}(r_0)}\right)^{\frac{1}{s}}\left(\int_M|f|^s\omega_1dA\right)^{\frac{1}{s}}\\&\lesssim\left(\int_{\mathbb{D}}|f|^s\omega_1dA\right)^{\frac{1}{s}}.\end{aligned}$$

由文 [11, 定理 1] 知

$$\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\in L^{\infty}(M_1)\ \text{且}\ \left\|\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\right\|_{L^{\infty}(M_1)}\lesssim\|I\|^{s\frac{p+q'}{pq'}}_{A^s_{\omega_{1,2}}\to L^{\frac{s(p+q')}{pq'}}(M,d\mu)}.$$

同理可得

$$\frac{\mu(S(\frac{r_0}{w}))}{\omega_1(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\in L^{\infty}(M_2)\ \text{且}\ \left\|\frac{\mu(S(\frac{r_0}{w}))}{\omega_1(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\right\|_{L^{\infty}(M_2)}\lesssim\|I\|^{s\frac{p+q'}{pq'}}_{A^s_{\omega_{1,2}}\to L^{\frac{s(p+q')}{pq'}}(M,d\mu)}.$$

下面证明 (4) $\Rightarrow$ (3). 若 (4) 成立, 设 $\{a_k\}$ 为 $M_1$ 的 Bergman 度量 $\rho(z,w)$ 的 $r$- 格点, $\{\frac{r_0}{b_k}\}$ 为 $M_2$ 的 Bergman 度量 $\rho(\frac{r_0}{z},\frac{r_0}{w})$ 的 $r$- 格点, 则

$$\begin{aligned}
\left(\int_M|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}&=\left(\int_{M_1}|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}+\left(\int_{M_2}|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\\
&\lesssim\left(\sum_{k=1}^{\infty}\int_{\Delta(a_k,r)}|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}+\left(\sum_{k=1}^{\infty}\int_{\Delta(\frac{r_0}{b_k},r)}|f|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\\
&\lesssim\left(\sum_{k=1}^{\infty}\sup_{z\in\Delta(a_k,r)}|f(z)|^{s\frac{p+q'}{pq'}}\mu(\Delta(a_k,r))\right)^{\frac{pq'}{s(p+q')}}\\
&\quad+\left(\sum_{k=1}^{\infty}\sup_{z\in\Delta(\frac{r_0}{b_k},r)}|f(z)|^{s\frac{p+q'}{pq'}}\mu\left(\Delta\left(\frac{r_0}{b_k},r\right)\right)\right)^{\frac{pq'}{s(p+q')}}\\
&\lesssim\left(\sum_{k=1}^{\infty}\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{\frac{p+q'}{pq'}}}\left(\int_{\Delta(a_k,3r)}|f(z)|^s\omega_1(z)dA(z)\right)^{\frac{p+q'}{pq'}}\right)^{\frac{pq'}{s(p+q')}}\\
&\quad+\left(\sum_{k=1}^{\infty}\frac{\mu(\Delta(\frac{r_0}{b_k},r))}{\omega_2(\Delta(\frac{r_0}{b_k},r))^{\frac{p+q'}{pq'}}}\left(\int_{\Delta(\frac{r_0}{b_k},3r)}|f(z)|^s\omega_2\left(\frac{r_0}{z}\right)dA(z)\right)^{\frac{p+q'}{pq'}}\right)^{\frac{pq'}{s(p+q')}}\\
&\lesssim\left\|\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))^{\frac{(p+q')}{pq'}}}\right\|^{\frac{pq'}{s(p+q')}}_{L^{\infty}(M_1)}\left(\sum_{k=1}^{\infty}\int_{\Delta(a_k,3r)}|f(z)|^s\omega_1(z)dA(z)\right)^{\frac{1}{s}}\\
&\quad+\left\|\frac{\mu(\Delta(\frac{r_0}{\cdot},r))}{\omega_2(\Delta(\frac{r_0}{\cdot},r))^{\frac{(p+q')}{pq'}}}\right\|^{\frac{pq'}{s(p+q')}}_{L^{\infty}(M_2)}\left(\sum_{k=1}^{\infty}\int_{\Delta(\frac{r_0}{b_k},3r)}|f(z)|^s\omega_2\left(\frac{r_0}{z}\right)dA(z)\right)^{\frac{1}{s}}\\
&\lesssim\left(\left\|\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))^{\frac{(p+q')}{pq'}}}\right\|^{\frac{pq'}{s(p+q')}}_{L^{\infty}(M_1)}+\left\|\frac{\mu(\Delta(\frac{r_0}{\cdot},r))}{\omega_2(\Delta(\frac{r_0}{\cdot},r))^{\frac{(p+q')}{pq'}}}\right\|^{\frac{pq'}{s(p+q')}}_{L^{\infty}(M_2)}\right)\|f\|_s.
\end{aligned}$$

故对 $0<s<\infty$, $\mu$ 是空间 $A^s_{\omega_{1,2}}$ 的 $\frac{s(p+q')}{pq'}$-Carleson 测度, 且

$$\|I\|^{s\frac{p+q'}{pq'}}_{A^s_{\omega_{1,2}}\to L^{\frac{s(p+q')}{pq'}}(M,d\mu)}\asymp\left\|\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))^{\frac{(p+q')}{pq'}}}\right\|_{L^{\infty}(M_1)}+\left\|\frac{\mu(\Delta(\frac{r_0}{\cdot},r))}{\omega_2(\Delta(\frac{r_0}{\cdot},r))^{\frac{(p+q')}{pq'}}}\right\|_{L^{\infty}(M_2)}.$$

现在证明 (1) $\Rightarrow$ (2). 若 $T_\mu:A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 有界, 则当 $w\in M_1$ 时,

$$\begin{aligned}
\frac{\tilde{T}_\mu(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}&=\int_M|b_w^{\omega_{1,2}}(z)|^2d\mu(z)\frac{1}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}\\
&=\int_M b_{p,w}^{\omega_{1,2}}(z)B_z^{\omega_{1,2}}(w)d\mu(z)\|B_w^{\omega_{1,2}}\|_2^{-2}\|B_w^{\omega_{1,2}}\|_p\frac{1}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim |T_\mu b_{p,w}^{\omega_{1,2}}(w)|\omega_1(S(w))^{\frac{1}{q}}\\
&\lesssim \left(\frac{1}{\omega_1(S(w))}\int_{\Delta(w,r)}|T_\mu b_{p,w}^{\omega_{1,2}}(\xi)|^q\omega_1(\xi)dA(\xi)\right)^{\frac{1}{q}}\omega_1(S(w))^{\frac{1}{q}}\\
&\lesssim \|T_\mu\|\,\|b_{p,w}^{\omega_{1,2}}\|_p \asymp \|T_\mu\|.
\end{aligned}
$$

同理, 当 $w\in M_2$ 时,

$$
\frac{\tilde{T}_\mu(w)}{\omega_1(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}} \lesssim \|T_\mu\|.
$$

若 $\frac{\tilde{T}_\mu(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}\in L^\infty(M_1)$, 则只需证 $\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\in L^\infty(M_1)$. 当 $w\in M_1$, $r>0$ 时,

$$
\begin{aligned}
\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}} &\lesssim \frac{1}{\omega_1(S(w))^{\frac{(p+q')}{pq'}-1}}\int_{\Delta(w,r)}|B_w^{\omega_{1,2}}(z)|d\mu(z)\\
&\lesssim \frac{1}{\omega_1(S(w))^{\frac{(p+q')}{pq'}-1}}\int_{\Delta(w,r)}|b_w^{\omega_{1,2}}(z)|^2 d\mu(z)\\
&\lesssim \frac{\widetilde{T_\mu}(w)}{\omega_1(S(w))^{\frac{(p+q')}{pq'}-1}}.
\end{aligned}
$$

同理可得, 若 $\frac{\tilde{T}_\mu(\frac{r_0}{w})}{\omega_1(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}}\in L^\infty(M_2)$, 则 $\frac{\mu(S(\frac{r_0}{w}))}{\omega_1(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\in L^\infty(M_2)$. 于是 (2) 可以推出 (4).

下证 (4) $\Rightarrow$ (1). 直接计算得

$$
\begin{aligned}
\int_M |f(z)|^p d\mu &= \int_{M_1}|f(z)|^p d\mu(z) + \int_{M_2}|f(z)|^p d\mu(z)\\
&\lesssim \int_{M_1} d\mu(z)\frac{1}{\omega_1(\Delta(z,r))}\int_{\Delta(z,r)}|f(\xi)|^p\omega_1(\xi)dA(\xi)\\
&\quad + \int_{M_2} d\mu(z)\frac{1}{\omega_2(\Delta(\frac{r_0}{z},r))}\int_{\Delta(\frac{r_0}{z},r)}|f(\xi)|^p\omega_2(\frac{r_0}{\xi})dA(\xi)\\
&\lesssim \int_{M_1}|f(\xi)|^p\omega_1(\xi)dA(\xi)\frac{1}{\omega_1(\Delta(\xi,r))}\int_{\Delta(\xi,2r)}d\mu(z)\\
&\quad + \int_{M_2}|f(\xi)|^p\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\frac{1}{\omega_2(\Delta(\frac{r_0}{\xi},r))}\int_{\Delta(\frac{r_0}{\xi},2r)}d\mu(z)\\
&\lesssim \int_{M_1}|f(\xi)|^p\omega_1(\xi)\frac{\mu(\Delta(\xi,2r))}{\omega_1(\Delta(\xi,2r))}dA(\xi)\\
&\quad + \int_{M_2}|f(\xi)|^p\omega_2\left(\Delta\left(\frac{r_0}{\xi},2r\right)\right)\frac{\mu(\Delta(\frac{r_0}{\xi},2r))}{\omega_2(\Delta(\frac{r_0}{\xi},2r))}dA(\xi).
\end{aligned}
$$

故

$$
\begin{aligned}
|T_\mu f(z)|^q\omega_{1,2}(z) &\lesssim \left(\int_M |f(w)B_w^{\omega_{1,2}}(z)|d\mu(w)\right)^q\omega_{1,2}(z)\\
&\lesssim \Bigg(\int_{M_1}|f(w)B_w^{\omega_{1,2}}(z)|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\omega_1(w)dA(w)\\
&\quad + \int_{M_2}|f(w)B_w^{\omega_{1,2}}(z)|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_2(\Delta(\frac{r_0}{w},r))}\omega_2\left(\frac{r_0}{w}\right)dA(w)\Bigg)^q\omega_{1,2}(z)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_{M_1}\left|f(w)\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q\omega_1(w)dA(w)\left(\int_{M_1}|B_w^{\omega_{1,2}}(z)|^{q'}\omega_1(w)dA(w)\right)^{\frac{q}{q'}}\omega_{1,2}(z)\\
&\quad+\int_{M_2}\left|f(w)\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\left(\int_{M_2}|B_w^{\omega_{1,2}}(z)|^{q'}\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)^{\frac{q}{q'}}\omega_{1,2}(z)\\
&\lesssim \left(\int_{M_1}\left|f(w)\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q\omega_1(w)dA(w)\right.\\
&\quad\left.+\int_{M_2}\left|f(w)\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)\|B_z^{\omega_{1,2}}\|^{\frac{q}{q'}}\omega_{1,2}(z).
\end{aligned}
$$

由引理 3.2, 定理 2.11, (4) 及 Fubini 定理, 得

$$
\begin{aligned}
\int_M|T_\mu f(z)|^q\omega_{1,2}(z)dA(z) &\lesssim \int_M\left(\int_{M_1}\left|f(w)\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q\omega_1(w)dA(w)\right.\\
&\quad\left.+\int_{M_2}\left|f(w)\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)\|B_z^{\omega_{1,2}}\|_{q'}^q\omega_{1,2}(z)dA(z)\\
&\lesssim \left(\int_{M_1}\left|f(w)\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q\omega_1(w)dA(w)\right.\\
&\quad\left.+\int_{M_2}\left|f(w)\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)\int_M\|B_z^{\omega_{1,2}}\|_q^q\omega_{1,2}(z)dA(z)\\
&\lesssim \left(\int_{M_1}\left|f(w)\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q\omega_1(w)dA(w)\right.\\
&\quad\left.+\int_{M_2}\left|f(w)\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)\\
&\quad\times\left(\int_{M_1}\frac{1}{\omega_1(z)}\omega_{1,2}(z)dA(z)+\int_{M_2}\frac{1}{\omega_2(\frac{r_0}{z})}\omega_{1,2}(z)dA(z)\right)\\
&\lesssim \|f\|_p^{q-p}\left(\int_{M_1}|f(w)|^p\left|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))^{\frac{p+q'}{pq'}}}\right|^q\omega_1(w)dA(w)\right.\\
&\quad\left.+\int_{M_2}|f(w)|^p\left|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))^{\frac{p+q'}{pq'}}}\right|^q\omega_2\left(\frac{r_0}{w}\right)dA(w)\right)\\
&\lesssim \|f\|_p^q\left(\left\|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))^{\frac{p+q'}{pq'}}}\right\|_{L^\infty(M_1)}+\left\|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))^{\frac{p+q'}{pq'}}}\right\|_{L^\infty(M_2)}\right)^q,
\end{aligned}
$$

于是 $T_\mu$ 有界且 $\|T_\mu\|\lesssim\|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))^{\frac{p+q'}{pq'}}}\|_{L^\infty(M_1)}+\|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))^{\frac{p+q'}{pq'}}}\|_{L^\infty(M_2)}$.

最后证明 (2) 可以推出 (4). 由定理 2.11 与引理 2.12, 对 $r>0$, $w\in M_1$, 得

$$
\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\lesssim\frac{\int_{\Delta(w,r)}|b_w^{\omega_{1,2}}(z)|^2\omega_1dA(z)}{\omega_1(\Delta(w,r))^{\frac{(p+q')}{pq'}-1}}\lesssim\frac{\widetilde{T}_\mu(w)}{\omega_1(\Delta(w,r))^{\frac{(p+q')}{pq'}-1}}.
$$

同理对 $r>0$, $w\in M_2$, 得

$$
\frac{\mu(S(\frac{r_0}{w}))}{\omega_2(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\lesssim\frac{\int_{\Delta(\frac{r_0}{w},r)}|b_w^{\omega_{1,2}}(z)|^2\omega_2(\frac{r_0}{z})dA(z)}{\omega_2(\Delta(\frac{r_0}{w},r))^{\frac{(p+q')}{pq'}-1}}\lesssim\frac{\widetilde{T}_\mu(w)}{\omega_2(\Delta(\frac{r_0}{w},r))^{\frac{(p+q')}{pq'}-1}}.
$$

证毕.

类似地, 可以得到 Toeplitz 算子紧性的等价刻画.

**定理 3.4**　若 $1<p\le q<\infty$, $\omega_1,\omega_2\in\mathscr{R}$ 且 $\mu$ 为 $M$ 上有限正 Borel 测度, 则下面条件等价:

(1) $T_\mu: A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 紧;

(2) 当 $|w|\to 1^-$, $\frac{\widetilde{T_\mu}(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}\to 0$, 且当 $|w|\to r_0^+$, $\frac{\widetilde{T_\mu}(w)}{\omega_2(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}}\to 0$, 其中 $q'$ 是 $q$ 的共轭数;

(3) 对 $0<s<\infty$ 中某个 (全部的) $s$, $\mu$ 是空间 $A^s_{\omega_{1,2}}$ 的 $\frac{s(p+q')}{pq'}$- 消失 Carleson 测度;

(4) 当 $|w|\to 1^-$, $\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\to 0$, 且当 $|w|\to r_0^+$, $\frac{\mu(S(\frac{r_0}{w}))}{\omega_2(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\to 0$.

**证明**　首先证明 (3) 推出 (4). 当 $s>0$, 若 $\{f_n\}\subset A^s_{\omega_1}(\mathbb{D})$ 有界且 $f_n$ 在 $\mathbb{D}$ 上内闭一致收敛于 0, 则 $\{f_n\}\subset A^s_{\omega_{1,2}}(M)$ 有界且 $f_n$ 在 $M$ 上内闭一致收敛于 0. 因为 $\mu$ 可以延拓为 $\mathbb{D}$ 上的测度, 使得支集为 $M$, 所以

$$\left(\int_{\mathbb{D}}|f_n|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\le\left(\int_M|f_n|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\to 0.$$

由文 [11, 定理 1] 知, 当 $|w|\to 1^-$, $\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}\to 0$. 同理, 当 $|w|\to r_0^+$, $\frac{\mu(S(\frac{r_0}{w}))}{\omega_1(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}\to 0$.

下面证明 (4) $\Rightarrow$ (3). 若 $\{f_n\}\subset A^s_{\omega_{1,2}}(M)$ 有界, $f_n$ 在 $M$ 上内闭一致收敛于 0, 且 (4) 成立. 设 $\{a_k\}$ 为 $M_1$ 的 Bergman 度量 $\rho(z,w)$ 的 $r$- 格点, $\{\frac{r_0}{b_k}\}$ 为 $M_2$ 的 Bergman 度量 $\rho(\frac{r_0}{z},\frac{r_0}{w})$ 的 $r$- 格点, 则对任意 $\varepsilon>0$, 存在 $K\in\mathbb{Z}^+$, 使得当 $k>K$ 时, 有

$$\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{\frac{p+q'}{pq'}}}<\varepsilon\quad\text{且}\quad\frac{\mu(\Delta(\frac{r_0}{b_k},r))}{\omega_2(\Delta(\frac{r_0}{b_k},r))^{\frac{p+q'}{pq'}}}<\varepsilon,$$

$$\begin{aligned}&\left(\int_M|f_n|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\\ &\lesssim\varepsilon^{\frac{pq'}{s(p+q')}}\|f_n\|_s+\left\|\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))^{\frac{(p+q')}{pq'}}}\right\|_{L^\infty(M_1)}^{\frac{pq'}{s(p+q')}}\left(\sum_{k=1}^K\int_{\Delta(a_k,3r)}|f_n(z)|^s\omega_1(z)dA(z)\right)^{\frac{1}{s}}\\ &\quad+\varepsilon^{\frac{pq'}{s(p+q')}}\|f_n\|_s+\left\|\frac{\mu(\Delta(\frac{r_0}{\cdot},r))}{\omega_1(\Delta(\frac{r_0}{\cdot},r))^{\frac{(p+q')}{pq'}}}\right\|_{L^\infty(M_2)}^{\frac{pq'}{s(p+q')}}\left(\sum_{k=1}^K\int_{\Delta(\frac{r_0}{b_k},3r)}|f_n(z)|^s\omega_2\left(\frac{r_0}{z}\right)dA(z)\right)^{\frac{1}{s}}.\end{aligned}$$

由 $f_n$ 在 $M$ 上内闭一致收敛于 0 知

$$\left(\int_M|f_n|^{s\frac{p+q'}{pq'}}d\mu\right)^{\frac{pq'}{s(p+q')}}\to 0.$$

现在来证明 (1) $\Rightarrow$ (2). 若 $T_\mu: A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 紧, 当 $|w|\to 1^-$ 时, 因 $b^{\omega_{1,2}}_{p,w}$ 在 $A^p_{\omega_{1,2}}(M)$ 中弱收敛于 0, 利用定理 3.3 中同样的分析得

$$\begin{aligned}\frac{\tilde{T}_\mu(w)}{\omega_1(S(w))^{\frac{1}{p}+\frac{1}{q'}-1}}&\lesssim\left(\frac{1}{\omega_1(S(w))}\int_{\Delta(w,r)}|T_\mu b^{\omega_{1,2}}_{p,w}(\xi)|^q\omega_1(\xi)dA(\xi)\right)^{\frac{1}{q}}\omega_1(S(w))^{\frac{1}{q}}\\ &\lesssim\|T_\mu b^{\omega_{1,2}}_{p,w}\|_q\to 0.\end{aligned}$$

同理可得, 当 $|w|\to r_0^+$ 时,

$$\frac{\tilde{T}_\mu(w)}{\omega_2(S(\frac{r_0}{w}))^{\frac{1}{p}+\frac{1}{q'}-1}}\to 0.$$

下证 (4) $\Rightarrow$ (1). 设 $\{f_n\}\subset A^p_{\omega_{1,2}}(M)$ 在 $A^p_{\omega_{1,2}}(M)$ 中弱收敛于 0, 则 $f_n$ 有界且在 $M$ 上内闭一致收敛于 0. 由 (4) 成立知对任意 $\varepsilon>0$, 存在 $r_1,r_2$, 使得当 $r_0<|w|<r_1$ 时, $\frac{\mu(S(\frac{r_0}{w}))}{\omega_2(S(\frac{r_0}{w}))^{\frac{(p+q')}{pq'}}}<\varepsilon$ 且当 $1>|w|>r_2$ 时, $\frac{\mu(S(w))}{\omega_1(S(w))^{\frac{(p+q')}{pq'}}}<\varepsilon$. 由定理 3.3 的证明得

$$\begin{aligned}\int_M|T_\mu f_n(z)|^q\omega_{1,2}(z)dA(z)\lesssim{}&\|f_n\|_p^{q-p}\varepsilon^q\int_{M_1}|f_n(w)|^p\omega_{1,2}(w)dA(w)\\&+\|f_n\|_p^{q-p}\left(\left\|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))^{\frac{p+q'}{pq'}}}\right\|_{L^\infty(M_1)}+\left\|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))^{\frac{p+q'}{pq'}}}\right\|_{L^\infty(M_2)}\right)^q\\&\times\int_{r_1<|w|<r_2}|f_n|^p\omega_{1,2}(w)dA(w),\end{aligned}$$

于是当 $n\to\infty$ 时, $\int_M|T_\mu f_n(z)|^q\omega_{1,2}(z)dA(z)\to 0$, 故 $T_\mu$ 为紧算子.

最后, 由定理 3.3 知若 (2) 成立, 则 (4) 显然成立. 证毕.

前面讨论了 Toeplitz 算子从小指标 $p$ 到大指标 $q$ 空间的有界性与紧性, 那么当 $p>q$ 时会有怎样的情形呢? 下面我们将会讨论这个问题. 在此之前需要证明一个引理.

**引理 3.5** 设 $\{a_k\}$ 为 $M_1$ 的 Bergman 度量 $\rho(z,w)$ 的某个 (任意) $r$- 格点, $\{\frac{r_0}{b_k}\}$ 为 $M_2$ 的 Bergman 度量 $\rho(\frac{r_0}{z},\frac{r_0}{w})$ 的某个 (任意) $r$- 格点. 若 $1\le p\le\infty$ 且 $\{\alpha_k\},\{\beta_k\}\in l^p$, 则当 $z\in M_1$时,

$$f(z)=\sum_{k=1}^\infty\alpha_k b^{\omega_{1,2}}_{p,a_k}(z)\in A^p_{\omega_{1,2}}\ \text{且}\ \|f\|_p\lesssim\|\{\alpha_k\}\|_{l^p};$$

当 $z\in M_2$ 时,

$$g(z)=\sum_{k=1}^\infty\beta_k b^{\omega_{1,2}}_{p,\frac{r_0}{b_k}}\left(\frac{r_0}{z}\right)\in A^p_{\omega_{1,2}}\ \text{且}\ \|g\|_p\lesssim\|\{\beta_k\}\|_{l^p},$$

其中 $1<p<\infty$.

**证明** 对 $r_0<r_1<|z|<r_2<1$, 有

$$\left|\sum_{k=1}^\infty\alpha_k b^{\omega_{1,2}}_{p,a_k}(z)\right|\lesssim\|\{\alpha_k\}\|_{l^p}\left(\sum_{k=1}^\infty\omega_1(\Delta(a_k,r))|B^{\omega_{1,2}}_{a_k}(z)|^{p'}\right)^{\frac{1}{p'}},$$

其中 $p'$ 是 $p$ 的对偶指标, 故 $f(z)$ 在 $M$ 上解析.

对任意 $h\in A^q_{\omega_{1,2}}$, 利用 Hölder 不等式与 $|h|$ 的次调和函数, 得

$$\begin{aligned}\left|\int_M h(z)\overline{f(z)}\omega_{1,2}dA(z)\right|&=\left|\sum_{k=1}^\infty\overline{\alpha_k}\frac{h(a_k)}{\|B^{\omega_{1,2}}_{a_k}\|_p}\right|\\&\lesssim\|\{\alpha_k\}\|_{l^p}\left(\sum_{k=1}^\infty\omega_1(\Delta(a_k,r))|h(a_k)|^{p'}\right)^{\frac{1}{p'}}\\&\lesssim\|\{\alpha_k\}\|_{l^p}\left(\sum_{k=1}^\infty\int_{\Delta(a_k,r)}|h(z)|^{p'}\omega_1(z)dA(z)\right)^{\frac{1}{p'}}\\&\lesssim\|\{\alpha_k\}\|_{l^p}\|h\|_{p'}.\end{aligned}$$

故 $f\in A^p_{\omega_{1,2}}$ 且 $\|f\|_p\lesssim\|\{\alpha_k\}\|_{l^p}$.

同理可得 $g\in A^p_{\omega_{1,2}}$ 且 $\|g\|_p\lesssim\|\{\beta_k\}\|_{l^p}$. 证毕.

**定理 3.6** 若 $1<q<p<\infty, \omega_1,\omega_2\in\mathscr{R}$ 且 $\mu$ 为 $M$上有限正 Borel 测度, 则以下条件等价:

(1) $T_\mu: A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 有界;

(2) $T_\mu: A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}$ 紧;

(3) $\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))}\in L^{\frac{pq}{p-q}}_{\omega_1}(M_1)$ 且 $\frac{\mu(\Delta(\frac{r_0}{\cdot},r))}{\omega_2(\Delta(\frac{r_0}{\cdot},r))}\in L^{\frac{pq}{p-q}}_{\omega_2}(M_2)$;

(4) 设 $\{a_k\}$ 为 $M_1$ 的 Bergman 度量 $\rho(z,w)$ 的某个 (任意) $r$- 格点, $\{\frac{r_0}{b_k}\}$ 为 $M_2$ 的 Bergman 度量 $\rho(\frac{r_0}{z},\frac{r_0}{w})$ 的某个 (任意) $r$- 格点, 则

$$\left\{\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))}\omega_1(\Delta(a_k,r))^{\frac{1}{q}-\frac{1}{p}}\right\}\in l^{\frac{pq}{p-q}} \text{ 且 } \left\{\frac{\mu(\Delta(\frac{r_0}{b_k},r))}{\omega_1(\Delta(\frac{r_0}{b_k},r))}(1-|a_k|^2)^{\frac{q}{p}-1}\right\}\in l^{\frac{pq}{p-q}};$$

(5) $\widetilde{T}_\mu(\cdot)\in L^{\frac{pq}{p-q}}_{\omega_1}(M_1)$ 且 $\widetilde{T}_\mu(\cdot)\in L^{\frac{pq}{p-q}}_{\omega_2}(M_2)$.

**证明** 由定理 2.11 与引理 2.12 可知, 对 $w\in M_1$, $r>0$, 有

$$\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\lesssim\frac{\int_{\Delta(w,r)}d\mu(z)}{\omega_1(\Delta(w,r))}\lesssim\int_{\Delta(w,r)}|b_w^{\omega_{1,2}}(z)|^2d\mu(z)\lesssim\widetilde{T}_\mu(w).$$

同理, 对 $w\in M_2$, $r>0$, 有

$$\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_2(\Delta(\frac{r_0}{w},r))}\lesssim\widetilde{T}_\mu(w).$$

故若 (5) 成立, 则 (3) 成立.

对 $f\in L^1_{\omega_1}(M_1)$, 由 Fubini 定理得

$$\begin{aligned}\|\widetilde{f}\|_{L^1_{\omega_1}(M_1)}&=\int_{M_1}\omega_1(w)dA(w)\left|\int_{M_1}|b_w^{\omega_{1,2}}(z)|^2f(z)\omega_1(z)dA(z)\right|\\&\le\int_{M_1}|f(z)|\omega_1(z)dA(z)\int_{M_1}|b_w^{\omega_{1,2}}(z)|^2\omega_1(w)dA(w)\\&\le\int_{M_1}|f(z)|\omega_1(z)dA(z),\end{aligned}$$

且显然有

$$\|\widetilde{f}\|_{L^\infty_{\omega_1}(M_1)}\lesssim\|f\|_{L^\infty_{\omega_1}(M_1)}.$$

于是由插值理论知对 $1\le t\le\infty$, 有 $\|\widetilde{f}\|_{L^t_{\omega_1}(M_1)}\lesssim\|f\|_{L^t_{\omega_1}(M_1)}$. 同理有 $\|\widetilde{f}\|_{L^t_{\omega_2}(M_2)}\lesssim\|f\|_{L^t_{\omega_2}(M_2)}$.

由定理 3.3 证明知, 对 $w\in M_1$, $r>0$, 有

$$\begin{aligned}\widetilde{T}_\mu(w)&=\int_M|b_w^{\omega_{1,2}}(z)|^2d\mu(z)\\&\le\int_{M_1}|b_w^{\omega_{1,2}}(z)|^2\frac{\mu(\Delta(z,r))}{\omega_1(\Delta(z,r))}\omega_1(z)dA(z)\int_{M_2}|b_w^{\omega_{1,2}}(z)|^2d\mu(z)\\&\le\left(\widetilde{\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))}}\right)(w)+\sup_{z\in M_2,\,w\in M_1}|b_w^{\omega_{1,2}}(z)|^2\mu(M_2)\\&\le\left(\widetilde{\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))}}\right)(w)+C.\end{aligned}$$

上面最后一个不等式是因为 $\sup_{z\in M_2,\,w\in M_1}|b_w^{\omega_{1,2}}(z)|^2<\infty$ 与 $\mu(M_2)$ 为常数.

于是若 (3) 成立, 则

$$\|\widetilde{T}_\mu(*)\|_{L^{\frac{pq}{p-q}}_{\omega_1}(M_1)} \le \left\|\left(\widetilde{\frac{\mu(\Delta(\cdot,r))}{\omega_1(\Delta(\cdot,r))}}\right)(*)\right\|_{L^{\frac{pq}{p-q}}_{\omega_1}(M_1)} + C \le \left\|\frac{\mu(\Delta(*,r))}{\omega_1(\Delta(*,r))}\right\|_{L^{\frac{pq}{p-q}}_{\omega_1}(M_1)} + C < \infty.$$

同理

$$\|\widetilde{T}_\mu(*)\|_{L^{\frac{pq}{p-q}}_{\omega_2}(M_2)} \le \left\|\frac{\mu(\Delta(\frac{r_0}{*},r))}{\omega_1(\Delta(\frac{r_0}{*},r))}\right\|_{L^{\frac{pq}{p-q}}_{\omega_2}(M_2)} + C < \infty.$$

从而 (5) 成立.

下证 (3) $\Rightarrow$ (2). 因 $\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))} \in L^{\frac{pq}{p-q}}_{\omega_1}(M_1)$ 且 $\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))} L^{\frac{pq}{p-q}}_{\omega_2}(M_2)$, 故对任意 $\varepsilon > 0$, 存在 $r_0 < r_1$, $r_2 < 1$, 使得

$$\int_{r_0<|w|<r_1} \left|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^{\frac{pq}{p-q}} \omega_2\left(\frac{r_0}{w}\right) dA(w) < \varepsilon \ \text{且}\ \int_{r_2<|w|<1} \left|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^{\frac{pq}{p-q}} \omega_1(w) dA(w) < \varepsilon.$$

若 $\{f_m\} \subset A^p_{\omega_{1,2}}(M)$ 有界且在 $A^p_{\omega_{1,2}}(M)$ 弱收敛于 0, 则 $f_m$ 在 $M$ 中内闭一致收敛于 0. 由定理 3.3 的证明可得, 对 $r > 0$ 有

$$\begin{aligned}
\|T_\mu f_m\|_q^q &= \int_M |T_\mu f_m(w)|^q \omega_{1,2}(w) dA(w) \\
&\lesssim \int_{M_1} \left| f_m(w) \frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^q \omega_1(w) dA(w) + \int_{M_2} \left| f_m(w) \frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_1(\Delta(\frac{r_0}{w},r))}\right|^q \omega_2\left(\frac{r_0}{w}\right) dA(w) \\
&\lesssim \left\|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right\|^q_{L^{\frac{pq}{p-q}}_{\omega_1}(M_1)} \left(\int_{\frac{1+r_0}{2}<|w|\le r_1} |f_m(w)|^p \omega_1(w) dA(w)\right)^{\frac{q}{p}} \\
&\quad + \|f_m\|_p^q \left(\int_{r_1<|w|<1} \left|\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right|^{\frac{pq}{p-q}} \omega_1(w) dA(w)\right)^{\frac{p-q}{p}} \\
&\quad + \left\|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_2(\Delta(\frac{r_0}{w},r))}\right\|^q_{L^{\frac{pq}{p-q}}_{\omega_2}(M_2)} \left(\int_{r_1<|w|\le\frac{1+r_0}{2}} |f_m(w)|^p \omega_2\left(\frac{r_0}{w}\right) dA(w)\right)^{\frac{q}{p}} \\
&\quad + \|f_m\|_p^q \left(\int_{r_0<|w|<r_1} \left|\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_2(\Delta(\frac{r_0}{w},r))}\right|^{\frac{pq}{p-q}} \omega_2\left(\frac{r_0}{w}\right) dA(w)\right)^{\frac{p-q}{p}},
\end{aligned}$$

于是当 $n \to \infty$ 时, $\|T_\mu f_m\|^q \to 0$, 故 $T_\mu$ 为紧算子.

对 $r > 0$, $\{\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))}\omega_1(\Delta(a_k,r))^{\frac{1}{q}-\frac{1}{p}}\} \in l^{\frac{pq}{p-q}}$, 有

$$\begin{aligned}
&\int_{M_1} \left(\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right)^{\frac{pq}{p-q}} \omega_1(w) dA(w) \\
&\quad \lesssim \sum_{k=1}^\infty \int_{\Delta(a_k,r)} \left(\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\right)^{\frac{pq}{p-q}} \omega_1(w) dA(w) \\
&\quad \lesssim \sum_{k=1}^\infty \omega_1(\Delta(a_k,r)) \left(\frac{\omega_1(\Delta(a_k,2r))}{\omega_1(\Delta(a_k,r))}\right)^{\frac{pq}{p-q}} \left(\frac{\mu(\Delta(a_k,2r))}{\omega_1(\Delta(a_k,2r))}\right)^{\frac{pq}{p-q}} \\
&\quad \lesssim \sum_{k=1}^\infty \omega_1(\Delta(a_k,r)) \left(\frac{\mu(\Delta(a_k,2r))}{\omega_1(\Delta(a_k,2r))}\right)^{\frac{pq}{p-q}} < \infty.
\end{aligned}$$

同理

$$\int_{M_2}\left(\frac{\mu(\Delta(\frac{r_0}{w},r))}{\omega_2(\Delta(\frac{r_0}{w},r))}\right)^{\frac{pq}{p-q}}\omega_2\left(\frac{r_0}{w}\right)dA(w)\lesssim\sum_{k=1}^{\infty}\omega_2\left(\Delta\left(\frac{r_0}{b_k},r\right)\right)\left(\frac{\mu(\Delta(\frac{r_0}{b_k},2r))}{\omega_2(\Delta(\frac{r_0}{b_k},2r))}\right)^{\frac{pq}{p-q}}<\infty.$$

于是 (4) $\Rightarrow$ (3).

对 $r>0$, 若 $\frac{\mu(\Delta(w,r))}{\omega_1(\Delta(w,r))}\in L^{\frac{pq}{p-q}}_{\omega_1}(M_1)$, 则

$$\begin{aligned}\sum_{k=1}^{\infty}\omega_1(\Delta(a_k,r))\left(\frac{\omega_1(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))}\right)^{\frac{pq}{p-q}}&\lesssim\sum_{k=1}^{\infty}\int_{\Delta(a_k,r)}\left(\frac{\omega_1(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))}\right)^{\frac{pq}{p-q}}\omega_1(w)dA(w)\\&\lesssim\sum_{k=1}^{\infty}\int_{\Delta(a_k,2r)}\left(\frac{\omega_1(\Delta(w,2r))}{\omega_1(\Delta(w,2r))}\right)^{\frac{pq}{p-q}}\omega_1(w)dA(w)\\&\lesssim\int_{M_1}\left(\frac{\omega_1(\Delta(w,2r))}{\omega_1(\Delta(w,2r))}\right)^{\frac{pq}{p-q}}\omega_1(w)dA(w)<\infty.\end{aligned}$$

同理

$$\sum_{k=1}^{\infty}\omega_2\left(\Delta\left(\frac{r_0}{b_k},r\right)\right)\left(\frac{\omega_2(\Delta(\frac{r_0}{b_k},r))}{\omega_2(\Delta(\frac{r_0}{b_k},r))}\right)^{\frac{pq}{p-q}}\lesssim\int_{M_1}\left(\frac{\omega_2(\Delta(\frac{r_0}{w},2r))}{\omega_2(\Delta(\frac{r_0}{w},2r))}\right)^{\frac{pq}{p-q}}\omega_2\left(\frac{r_0}{w}\right)dA(w)<\infty.$$

故 (3) $\Rightarrow$ (4).

下面证明 (1) $\Rightarrow$ (4). 若 $T_\mu$ 有界, 由引理 3.5 知, 对 $M_1$ 的格点 $\{a_k\}$, 当 $\{\alpha_k\}\in l^p$ 时,

$$\left\|T_\mu\left(\sum_{k=1}^{\infty}\alpha_k b^{\omega_{1,2}}_{p,a_k}(z)\right)\right\|_q^q\lesssim\|T_\mu\|^q_{A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}}\|\{\alpha_k\}\|^q_{l^p}.$$

于是, 由 Khintchine 不等式可知, 对 $r>0$ 有

$$\begin{aligned}\|T_\mu\|^q_{A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}}\|\{\alpha_k\}\|^q_{l^p}&\gtrsim\int_M\left(\sum_{k=1}^{\infty}|\alpha_k|^2|T_\mu b^{\omega_{1,2}}_{p,a_k}(z)|^2\right)^{\frac{q}{2}}\omega_{1,2}(z)dA(z)\\&\gtrsim\sum_{k=1}^{\infty}|\alpha_k|^q\int_{\Delta(a_k,r)}|T_\mu b^{\omega_{1,2}}_{p,a_k}(z)|^q\omega_1(z)dA(z).\end{aligned}$$

进一步地, 利用 $|T_\mu b^{\omega_{1,2}}_{p,a_k}(z)|^q$ 的次调和性质可得

$$\begin{aligned}\int_{\Delta(a_k,r)}|T_\mu b^{\omega_{1,2}}_{p,a_k}(z)|^q\omega_1(z)dA(z)&\gtrsim\omega_1(\Delta(a_k,r))|T_\mu b^{\omega_{1,2}}_{p,a_k}(a_k)|^q\\&\gtrsim\frac{\omega_1(\Delta(a_k,r))}{\|B^{\omega_{1,2}}_{a_k}\|^q_p}\left(\int_M|B^{\omega_{1,2}}_{a_k}(\zeta)|^2d\mu\right)^q\\&\gtrsim\frac{\omega_1(\Delta(a_k,r))}{\|B^{\omega_{1,2}}_{a_k}\|^q_p}\left(\int_{\Delta(a_k,r)}|B^{\omega_{1,2}}_{a_k}(\zeta)|^2d\mu\right)^q\\&\gtrsim\frac{\omega_1(\Delta(a_k,r))}{\|B^{\omega_{1,2}}_{a_k}\|^q_p}\mu(\Delta(a_k,r))^q|B^{\omega_{1,2}}_{a_k}(a_k)|^{2q}\\&\asymp\left(\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{1+\frac{1}{p}-\frac{1}{q}}}\right)^q,\end{aligned}$$

于是有

$$\sum_{k=1}^{\infty}|\alpha_k|^q\left(\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{1+\frac{1}{p}-\frac{1}{q}}}\right)^q\lesssim\|T_\mu\|_{A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}}\|\{\alpha_k\}\|^q_{l^p}.$$

故由对偶理论可得

$$\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{1+\frac{1}{p}-\frac{1}{q}}}\in l^{\frac{pq}{p-q}},$$

从而

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))}\right)^{\frac{pq}{p-q}}\omega_1(\Delta(a_k,r))\\&=\sum_{k=1}^{\infty}\left(\frac{\mu(\Delta(a_k,r))}{\omega_1(\Delta(a_k,r))^{1+\frac{1}{p}-\frac{1}{q}}}\right)^{\frac{pq}{p-q}}\lesssim\|T_\mu\|^{\frac{pq}{p-q}}_{A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}}.\end{aligned}$$

同理

$$\sum_{k=1}^{\infty}\left(\frac{\mu(\Delta(\frac{r_0}{b_k},r))}{\omega_2(\Delta(\frac{r_0}{b_k},r))}\right)^{\frac{pq}{p-q}}\omega_2\left(\Delta\left(\frac{r_0}{b_k},r\right)\right)\lesssim\|T_\mu\|^{\frac{pq}{p-q}}_{A^p_{\omega_{1,2}}\to A^q_{\omega_{1,2}}}.$$

证毕.

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