第63卷第5期 数学学报中文版 Vol.63, No.5
2020年9月 ACTA MATHEMATICA SINICA, CHINESE SERIES Sep., 2020

文章编号: 0583-1431(2020)05-0427-16 **文献标识码:** A

具有一般齐次核多维的半离散 Hardy–Hilbert 型不等式

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摘 要 利用权函数、转换公式和实分析技巧, 给出一个具有一般齐次核和最佳常数因子的多维半离散 Hardy–Hilbert 型不等式, 它是一个已知结果的推广. 此外, 还讨论了等价形式、算子表示以及几种特殊应用例子.

关键词 半离散 Hardy–Hilbert 型不等式; 权函数; 等价形式; 算子; 范数
MR(2010) 主题分类 26D15
中图分类 O178

An Extended Multidimensional Half-discrete Hardy–Hilbert-type Inequality with Homogeneous Kernel

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Abstract By the use of the weight functions, the transfer formula and the technique of real analysis, an extended multidimensional half-discrete Hardy–Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of a published result. Moreover, the equivalent forms, a few particular cases and the operator expressions with some examples are considered.

Keywords half-discrete Hardy–Hilbert-type inequality; weight function; equivalent form; operator; norm
MR(2010) Subject Classification 26D15
Chinese Library Classification O178

收稿日期: 2019-07-17; 接受日期: 2019-12-03
基金项目: 国家自然科学基金资助项目 (61772140); 广州市科技规划项目 (201707010229)
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1 引言

设 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), g(y)\geq 0$, $f\in L^p(\mathbb{R}_+)$, $g\in L^q(\mathbb{R}_+)$, 满足

$$\|f\|_p=\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}>0$$

及 $\|g\|_q>0$, 则我们有如下 Hardy–Hilbert 积分不等式 [3]:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1.1}$$

这里, 常数因子 $\frac{\pi}{\sin(\pi/p)}$ 是最佳值. 设 $a_m, b_n\geq 0$, $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, 满足

$$\|a\|_p=\left(\sum_{m=1}^\infty a_m^p\right)^{\frac{1}{p}}>0$$

及 $\|b\|_q>0$, 我们还有上述积分不等式的具有相同最佳常数因子 $\frac{\pi}{\sin(\pi/p)}$ 的离散形式:

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{1.2}$$

(1.1) 和 (1.2) 是分析学及相关领域的重要不等式 [3, 13, 22–24, 26].

1998 年, 通过引入独立参数 $\lambda\in(0,1]$, 对于 $p=q=2$, 杨 [25] 给出了式 (1.1) 的一个推广. 2009–2011 年, 杨 [22, 23] 给出了式 (1.1) 和 (1.2) 的如下推广: 设 $\lambda_1,\lambda_2\in\mathbf{R}=(-\infty,\infty)$, $\lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)$ 是一个 $-\lambda$ 齐次非负函数, 满足

$$k_\lambda(tx,ty)=t^{-\lambda}k_\lambda(x,y)\quad(t,x,y>0),$$
$$k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt\in\mathbb{R}_+=(0,\infty),$$

又设 $\phi(x)=x^{p(1-\lambda_1)-1}$, $\psi(y)=y^{q(1-\lambda_2)-1}$, $f(x)$, $g(y)\geq 0$,

$$f\in L_{p,\phi}(\mathbb{R}_+)=\left\{f;\|f\|_{p,\phi}:=\left\{\int_0^\infty\phi(x)|f(x)|^pdx\right\}^{\frac{1}{p}}<\infty\right\},$$

$g\in L_{q,\psi}(\mathbb{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi}>0$, 则有

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy<k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{1.3}$$

这里, 常数因子 $k(\lambda_1)$ 是最佳值. 此外, 如果 $k_\lambda(x,y)$ 是有限的并且 $k_\lambda(x,y)x^{\lambda_1-1}$ $(k_\lambda(x,y)y^{\lambda_2-1})$ 关于 $x>0$ $(y>0)$ 递减, 那么对 $a_{m,}b_n\geq 0$,

$$a\in l_{p,\phi}=\left\{a;\|a\|_{p,\phi}:=\left\{\sum_{n=1}^\infty\phi(n)|a_n|^p\right\}^{\frac{1}{p}}<\infty\right\},$$

$b=\{b_n\}_{n=1}^\infty\in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi}>0$, 我们有

$$\sum_{m=1}^\infty\sum_{n=1}^\infty k_\lambda(m,n)a_mb_n<k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{1.4}$$

这里, 常数因子 $k(\lambda_1)$ 是最佳值. 显然, 若取 $\lambda=1$, $k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, 则式 (1.3) 变为式 (1.1), 同时式 (1.4) 变为式 (1.2). 文 [1, 2, 4–7, 10–12, 17, 21, 28, 30, 33–35, 38, 42] 给出了多重积分或离散的 Hilbert 型不等式的其他一些结果.

关于非齐次核的半离散 Hilbert 型不等式, Hardy 等在其专著 [3, 定理 351] 给出了一些结果. 但没有证明其常数因子的最佳性. 而杨 [18] 通过引入中间变量给出了一个核为 $\frac{1}{(1+nx)^{\lambda}}$ 的结果, 并证明其常数因子是最佳的. 2011 年, 杨 [19] 给出了如下具有最佳常数因子 $B(\lambda_1, \lambda_2)$ 的半离散的 Hardy–Hilbert 不等式:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} dx < B(\lambda_1, \lambda_2) \|f\|_{p,\phi} \|a\|_{q,\psi}, \tag{1.5}$$

这里, $\lambda_1, \lambda_2 > 0$, $\lambda_2 \le 1$, $\lambda_1 + \lambda_2 = \lambda$,

$$B(u,v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u,v > 0)$$

是 beta 函数. 钟等人 [36, 37, 39−43] 研究了几个具有特殊核的半离散 Hilbert 型不等式.

杨和陈 [29] 应用权函数的方法, 通过在离散和积分 Hilbert 型不等式的核添加条件, 建立了一个带一般 $-\lambda \in \mathbb{R}$ 齐次核的具有最佳常数因子 $k(\lambda_1)$ 的半离散 Hilbert 型不等式:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x,n) a_n dx < k(\lambda_1) \|f\|_{p,\phi} \|a\|_{q,\psi}, \tag{1.6}$$

它是式 (1.5) 的一个推广. 杨 [20] 还给出了一个带一般非齐次核的具有最佳常数因子的半离散 Hilbert 型不等式. 2013 至 2014 年间, Micheal 和杨 [15, 16] 给出了两个带特殊非齐次核的多维半离散 Hilbert 型不等式.

注 1.1 (1) 近二十年来, 出现了许多具有应用价值的离散、半离散和积分的 Hilbert 型不等式. 特别是 2009–2014 年期间被证明的一些新结果. 其中包括离散、半离散和积分的 Hilbert 型不等式的许多改进、推广和一般化, 涉及许多特殊函数如贝塔、伽玛、超几何、三角、双曲、泽塔、伯努利函数, 以及伯努利数和欧拉常数等.

(2) 杨的六本书中 (见 [22–24, 26, 27, 32]) 给出了具有一般齐次实数核和联系两对共轭指数的 Hilbert 型算子以及相关不等式的许多新结果. 这些研究专著包含了离散、半离散和积分类型的算子和不等式的最新发展, 并给出了证明、例子和应用。

本文利用权函数、转换公式和实分析技巧, 给出一个具有一般齐次核和最佳常数因子的多维半离散 Hardy–Hilbert 型不等式, 它是 (1.6) 的推广. 此外, 还讨论了等价形式、算子表示以及几种特殊应用例子.

2 一些引理

对 $\mu_i^{(k)} > 0$ $(k = 1, \dots, i_0;\ i = 1, \dots, m)$, $\upsilon_j^{(l)} > 0$ $(l = 1, \dots, j_0;\ j=1, \dots, n)$, 我们置

$$V_n^{(l)} := \sum_{j=1}^{n} \upsilon_j^{(l)} \quad (l = 1, \dots, j_0),$$

$$V_n = (V_n^{(1)}, \dots, V_n^{(j_0)}) \quad (n \in \mathbf{N} = \{1,2,\dots\}). \tag{2.1}$$

再设 $\mu_i(t) := \mu_m^{(i)}$, $t \in (m-1, m]$ $(m \in \mathbb{N})$; $\upsilon_j(t) := \upsilon_n^{(j)}$, $t \in (n-1, n]$ $(n \in \mathbb{N})$, 以及

$$U_i(x) := \int_0^{x} \mu_i(t) dt \quad (i = 1, \dots, i_0), \tag{2.2}$$

$$V_j(y) := \int_0^{y} \upsilon_j(t) dt \quad (j = 1, \dots, j_0), \tag{2.3}$$

$$U(x):=(U_1(x),\dots,U_{i_0}(x)),$$
$$V(y):=(V_1(y),\dots,V_{j_0}(y))\ \ (x,y\geq 0), \tag{2.4}$$

显然有

$$V_j(n)=V_n^{(j)}\ \ (j=1,\dots,j_0;\,n\in\mathbb{N}),$$

并且, 当 $x\in(m-1,m)$ 时, $U_i'(x)=\mu_i(x)=\mu_m^{(i)}$ $(i=1,\dots,i_0;\,m\in\mathbb{N})$; 当 $y\in(n-1,n)$ 时, $V_j'(y)=\upsilon_j(y)=\upsilon_n^{(j)}$ $(j=1,\dots,j_0;\,n\in\mathbb{N})$, 且有

$$dU(x)=\sum_{i=1}^{i_0}\mu_i(x)dx\ \ (x\in\mathbb{R}_+^{i_0}),\quad dV(y)=\sum_{j=1}^{j_0}\upsilon_j(y)dy\ \ (y\in\mathbb{R}_+^{j_0}).$$

引理 2.1 [31] 设 $g(t)(>0)$ 在 $\mathbb{R}_+$ 单调递减, 且在 $[n_0,\infty)$ $(n_0\in\mathbb{N})$ 严格递减, 满足

$$\int_0^\infty g(t)dt\in\mathbb{R}_+,$$

则有

$$\int_1^\infty g(t)dt<\sum_{n=1}^\infty g(n)<\int_0^\infty g(t)dt. \tag{2.5}$$

引理 2.2 设 $i_0\in\mathbb{N}$, α, $M>0$, $\Psi(u)$ 是 $(0,1]$ 上的非负可测函数, 且

$$D_M:=\left\{x\in\mathbb{R}_+^{i_0};\,u=\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\leq 1\right\}, \tag{2.6}$$

则有如下转换公式 [24]:

$$\int\cdots\int_{D_M}\Psi\left(\sum_{i=1}^{i_0}\left(\frac{x_i}{M}\right)^\alpha\right)dx_1\cdots dx_{i_0}=\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\Psi(u)u^{\frac{i_0}{\alpha}-1}du. \tag{2.7}$$

引理 2.3 对 $i_0,j_0\in\mathbb{N}$, $\upsilon_n^{(l)}\geq\upsilon_{n+1}^{(l)}$ $(n\in\mathbb{N};\,l=1,\dots,j_0)$, $\alpha,\beta,\varepsilon>0$,

$$b=\min_{1\leq i\leq i_0,\,1\leq j\leq j_0}\{\mu_1^{(i)},\upsilon_1^{(j)}\},\quad [1,\infty)^{i_0}:=\{x\in\mathbb{R}_+^{i_0};x_i\geq 1\,(i=1,\dots,i_0)\},$$

我们有

$$\int_{[1,\infty)^{i_0}}\|U(x)\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx\leq\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}, \tag{2.8}$$

$$\sum_n\|V_n\|_\beta^{-j_0-\varepsilon}\prod_{k=1}^{j_0}\upsilon_n^{(k)}\leq\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+O(1), \tag{2.9}$$

其中, $\sum_n=\sum_{n_{j_0}}^\infty\cdots\sum_{n_1}^\infty$,

$$\|x\|_\alpha=\left(\sum_{i=1}^{i_0}x_i^\alpha\right)^{\frac{1}{\alpha}}\ \ (x\in\mathbb{R}_+^{i_0}),\quad \|y\|_\beta=\left(\sum_{j=1}^{j_0}y_j^\beta\right)^{\frac{1}{\beta}}\ \ (y\in\mathbb{R}_+^{j_0}).$$

证明 设 $v=U(x)$, 则有

$$\int_{[1,\infty)^{i_0}}\|U(x)\|_\alpha^{-i_0-\varepsilon}\prod_{k=1}^{i_0}\mu_k(x)dx=\int_{\{v\in\mathbb{R}_+^{i_0};v_i\geq\mu_1^{(i)}\}}\frac{dv}{\|v\|_\alpha^{i_0+\varepsilon}}\leq\int_{[b,\infty)^{i_0}}\frac{dv}{\|v\|_\alpha^{i_0+\varepsilon}}.$$

对 $M > bi_0^{1/\alpha}$, 由 (2.7) 得到

$$\begin{aligned}\int_{[b,\infty)^{i_0}} \frac{dv}{\|v\|_\alpha^{i_0+\varepsilon}} &= \lim_{M\to\infty} \int_{\{v\in\mathbb{R}_+^{i_0};\frac{b^\alpha i_0}{M^\alpha}<\sum_{i=1}^{i_0}(\frac{v_i}{M})^\alpha\le 1\}} \frac{dv_1\cdots dv_{i_0}}{\{M[\sum_{i=1}^{i_0}(\frac{v_i}{M})^\alpha]^{\frac{1}{\alpha}}\}^{i_0+\varepsilon}} \\ &= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{b^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}} du \\ &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.\end{aligned}$$

因此式 (2.8) 成立.

我们有

$$\begin{aligned}&\sum_n \|V_n\|_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \le H_0 + \sum_{i=1}^{j_0} H_i, \\ &H_0 := \sum_{\{n\in\mathbb{N}^{j_0};\, n_j\ge 2\}} \|V_n\|_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)}, \\ &H_i := \sum_{\{n\in\mathbb{N}^{j_0};\, n_i=1;\, n_j\ge 1 (j\ne i)\}} \|V_n\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)}.\end{aligned}$$

与我们得到式 (2.8) 的方法相同, 由式 (2.5) 有

$$\begin{aligned}0 < H_0 &= \sum_{\{n\in\mathbb{N}^{j_0};\, n_j\ge 2\}} \int_{\{y\in\mathbb{N}^{j_0};\, n_j-1\le y_j<n_j\}} \|V(n)\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)} dy \\ &< \sum_{\{n\in\mathbb{N}^{j_0};\, n_j\ge 2\}} \int_{\{y\in\mathbb{R}_+^{j_0};\, n_j-1\le y_j<n_j\}} \|V(y)\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_l(y) dy \\ &= \int_{[1,\infty)^{j_0}} \|V(y)\|_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_l(y) dy \quad (v = V(y)) \\ &\le \int_{[b,\infty)^{j_0}} \|v\|_\beta^{-j_0-\varepsilon} dv \\ &= \frac{\Gamma^{i_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\alpha}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}.\end{aligned}$$

不失一般性, 我们估计 H_{j_0}. 如果 $j_0 = 1$, 则

$$0 < H_{j_0} = (\upsilon_1^{(1)})^{-1-\varepsilon}\upsilon_1^{(1)} = (\upsilon_1^{(1)})^{-\varepsilon} < \infty;$$

如果 $j_0 \ge 2$, 则有

$$\begin{aligned}H_{j_0} &= \sum_{\{n\in\mathbb{N}^{j_0-1}\}} \int_{\{y\in\mathbb{R}_+^{j_0-1};\, n_j-1<y_j\le n_j\}} \frac{\upsilon_1^{(j_0)} \prod_{l=1}^{j_0-1} \upsilon_l(y) dy}{[(\upsilon_1^{(j_0)})^\beta + \sum_{j=1}^{j_0-1}(V_n^{(j)})^\beta]^{\frac{j_0+\varepsilon}{\beta}}} \\ &\le \upsilon_1^{(j_0)} \int_{\mathbb{R}_+^{j_0-1}} \frac{\prod_{l=1}^{j_0-1} \upsilon_l(y)}{[(\upsilon_1^{(j_0)})^\beta + \sum_{j=1}^{j_0-1}(V^{(j)}(y))^\beta]^{\frac{1}{\beta}(j_0+\varepsilon)}} dy.\end{aligned}$$

设 $\upsilon=V(y)=(V_1(y),\ldots,V_{j_0-1}(y))$, 由式 (2.7) 有

$$\begin{aligned}0<H_{j_0}&\le \upsilon_1^{(j_0)}\int_{\mathbb{R}_+^{j_0-1}}\frac{1}{[(\upsilon_1^{(j_0)})^\beta+\sum_{j=1}^{j_0-1}\upsilon_j^\beta]^{\frac{1}{\beta}(j_0+\varepsilon)}}d\upsilon\\&=\upsilon_1^{(j_0)}\lim_{M\to\infty}\frac{M^{j_0-1}\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})}\int_0^1\frac{u^{\frac{j_0-1}{\beta}-1}du}{[(\upsilon_1^{(j_0)})^\beta+M^\beta u]^{\frac{1}{\beta}(j_0+\varepsilon)}}\\&\xlongequal{t=\frac{M^\beta u}{(\upsilon_1^{(j_0)})^\beta}}(\upsilon_1^{(j_0)})^{-\varepsilon}\frac{\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})}\int_0^\infty\frac{t^{\frac{j_0-1}{\beta}-1}}{(1+t)^{\frac{1}{\beta}(j_0+\varepsilon)}}dt\\&=(\upsilon_1^{(j_0)})^{-\varepsilon}\frac{\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})}B\left(\frac{j_0-1}{\beta},\frac{1+\varepsilon}{\beta}\right)<\infty.\end{aligned}$$

因此, 我们得到

$$\sum_n\|V_n\|_\beta^{-j_0-\varepsilon}\prod_{k=1}^{j_0}\upsilon_n^{(k)}\le\frac{\Gamma^{i_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\alpha}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\sum_{i=1}^{j_0}O_i(1),$$

即式 (2.9) 成立. 证毕.

定义 2.4 设 $i_0,j_0\in\mathbf{N},\alpha,\beta>0$, $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)$ 是一个 $-\lambda$ 齐次非负函数, 若对任意固定的 $x>0$, $k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}$ 对变量 $y\in\mathbb{R}_+$ 递减, 且在区间 $(b_x,\infty)\subset(0,\infty)$ 严格递减,

$$k(\lambda_1)=\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du\in\mathbb{R}_+,$$

定义如下两个权函数 $w(\lambda_1,n)$ $(n\in\mathbb{N}^{j_0})$ 和 $W(\lambda_2,x)$ $(x\in\mathbb{R}_+^{i_0})$:

$$w(\lambda_1,n):=\int_{\mathbb{R}_+^{i_0}}k_\lambda(\|U(x)\|_\alpha,\|V_n\|_\beta)\frac{\|V_n\|_\beta^{\lambda_2}}{\|U(x)\|_\alpha^{i_0-\lambda_1}}\prod_{k=1}^{i_0}\mu_k(x)dx,\tag{2.10}$$

$$W(\lambda_2,x):=\sum_n k_\lambda(\|U(x)\|_\alpha,\|V_n\|_\beta)\frac{\|U(x)\|_\alpha^{\lambda_1}}{\|V_n\|_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}\upsilon_n^{(l)}.\tag{2.11}$$

例 2.5 对 $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1+\lambda_2=\lambda$, $\lambda_1+\eta>0$, $0<\lambda_2+\eta\le j_0$, 设

$$k_\lambda(x,y)=\frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}\quad(x,y>0),$$

则对任意固定的 $x>0$,

$$k_\lambda(x,y)\frac{1}{y^{j_0-\lambda_2}}=\begin{cases}\dfrac{1}{x^{\lambda+\eta}y^{j_0-\lambda_2-\eta}}, & 0<y<x,\\[2mm] \dfrac{x^\eta}{y^{j_0+\lambda_1+\eta}}, & y\ge x\end{cases}$$

对变量 $y\in\mathbb{R}_+$ 递减, 且在区间 $([x]+1,\infty)\subset(0,\infty)$ 严格递减. 以及有

$$\begin{aligned}k(\lambda_1)&=\int_0^\infty\frac{(\min\{u,1\})^\eta}{(\max\{u,1\})^{\lambda+\eta}}\frac{1}{u^{1-\lambda_1}}du\\&=\int_0^1\frac{u^\eta}{u^{1-\lambda_1}}du+\int_1^\infty\frac{1}{u^{\lambda+\eta}}\frac{1}{u^{1-\lambda_1}}du\\&=\frac{\lambda+2\eta}{(\lambda_1+\eta)(\lambda_2+\eta)}\in\mathbb{R}_+.\end{aligned}$$

注 2.6 设 $b,\beta>0$, 有

$$\frac{d}{dy}(b+y^{\beta})^{\frac{1}{\beta}}=(b+y^{\beta})^{\frac{1}{\beta}-1}y^{\beta-1}>0\ \ (y>0).$$

因此, 按定义 2.4 条件, 对 $n_j-1<y_j<n_j\,(j=1,\dots,j_0;\,n\in\mathbb{N}^{j_0})$, 有

$$\|V(n)\|_{\beta}>\|V(y)\|_{\beta}$$

和

$$\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V(n)\|_{\beta})}{\|V(n)\|_{\beta}^{j_0-\lambda_2}}<\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V(y)\|_{\beta})}{\|V(y)\|_{\beta}^{j_0-\lambda_2}};$$

对 $n_j<y_j<n_j+1\ (j=1,\dots,j_0;\,n\in\mathbb{N}^{j_0})$, $\frac{\varepsilon}{q}>0$, 则有

$$\|V(n)\|_{\beta}<\|V(y)\|_{\beta}$$

和

$$\begin{aligned}\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V_n\|_{\beta})}{\|V_n\|_{\beta}^{j_0-\lambda_2+\frac{\varepsilon}{q}}}&=\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V(n)\|_{\beta})}{\|V(n)\|_{\beta}^{j_0-\lambda_2}}\frac{1}{\|V(n)\|_{\beta}^{\frac{\varepsilon}{q}}}\\&>\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V(y)\|_{\beta})}{\|V(y)\|_{\beta}^{j_0-\lambda_2}}\frac{1}{\|V(y)\|_{\beta}^{\frac{\varepsilon}{q}}}\\&=\frac{k_{\lambda}(\|U(x)\|_{\alpha},\|V(y)\|_{\beta})}{\|V(y)\|_{\beta}^{j_0-\lambda_2+\frac{\varepsilon}{q}}}.\end{aligned}\tag{2.12}$$

引理 2.7 在定义 2.4 的条件下, 我们有

$$w(\lambda_1,n)\le K_{\alpha}(\lambda_1)\ \ (n\in\mathbb{N}^{j_0}),\tag{2.13}$$

$$W(\lambda_2,x)<K_{\beta}(\lambda_1)\ \ (x\in\mathbb{R}_{+}^{i_0}),\tag{2.14}$$

其中

$$K_{\beta}(\lambda_1)=\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}k(\lambda_1),\quad K_{\alpha}(\lambda_1)=\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1);\tag{2.15}$$

证明 设 $v=\frac{U(x)}{\|V_n\|_{\beta}}$, 因 $U_k(\infty)\le\infty\ (k=1,\dots,i_0)$, 我们有

$$w(\lambda_1,n)\le\int_{\mathbb{R}_{+}^{i_0}}k_{\lambda}(\|v\|_{\alpha},1)\frac{1}{v^{i_0-\lambda_1}}dv.$$

由 (2.7) 得到

$$\begin{aligned}\int_{\mathbb{R}_{+}^{i_0}}k_{\lambda}(\|v\|_{\alpha},1)\frac{dv}{v^{i_0-\lambda_1}}&=\lim_{M\to\infty}\int_{D_M}\frac{k_{\lambda}(M[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^{\alpha}]^{\frac{1}{\alpha}},1)dv_1\cdots dv_{i_0}}{\{M[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^{\alpha}]^{\frac{1}{\alpha}}\}^{i_0-\lambda_1}}\\&=\lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\frac{k_{\lambda}(Mu^{\frac{1}{\alpha}},1)u^{\frac{i_0}{\alpha}-1}du}{(Mu^{1/\alpha})^{i_0-\lambda_1}}\\&=\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^{\infty}k_{\lambda}(v,1)v^{\lambda_1-1}dv\\&=\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1).\end{aligned}$$

因此, 式 (2.13) 成立.

由式 (2.5), 式 (2.7) 和注 2.6, 我们有

$$\begin{aligned}
W(\lambda_2,x) &= \sum_n \int_{\{y\in\mathbb{R}_+^{j_0};\, n_j-1<y_j\le n_j\}} k_\lambda(\|U(x)\|_\alpha,\|V(n)\|_\beta)\frac{\|U(x)\|_\alpha^{\lambda_1}}{\|V(n)\|_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}v_l(y)dy \\
&< \sum_n \int_{\{y\in\mathbb{R}_+^{j_0};\, n_j-1<y_j\le n_j\}} k_\lambda(\|U(x)\|_\alpha,\|V(y)\|_\beta)\frac{\|U(x)\|_\alpha^{\lambda_1}}{\|V(y)\|_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}v_l(y)dy \\
&= \int_{\mathbb{R}_+^{j_0}} k_\lambda(\|U(x)\|_\alpha,\|V(y)\|_\beta)\frac{\|U(x)\|_\alpha^{\lambda_1}}{\|V(y)\|_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}v_l(y)dy\,(v=V(y)) \\
&\le \int_{\mathbb{R}_+^{j_0}} k_\lambda(\|U(x)\|_\alpha,\|v\|_\beta)\frac{\|U(x)\|_\alpha^{\lambda_1}}{\|v\|_\beta^{j_0-\lambda_2}}dv \\
&= \lim_{M\to\infty}\int_{\left\{v\in\mathbb{R}_+^{j_0};\sum_{j=1}^{j_0}\left(\frac{v_j}{M}\right)^\alpha\le 1\right\}} k_\lambda\left(\|U(x)\|_\alpha, M\left[\sum_{j=1}^{j_0}\left(\frac{v_j}{M}\right)^\beta\right]^{1/\beta}\right) \\
&\quad\times\frac{\|U(x)\|_\alpha^{\lambda_1}}{\{M[\sum_{j=1}^{j_0}(\frac{v_j}{M})^\beta]^{\frac{1}{\beta}}\}^{j_0-\lambda_2}}dv_1\dots dv_{j_0} \\
&= \lim_{M\to\infty}\frac{M^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0}\Gamma(\frac{j_0}{\beta})}\int_0^1 k_\lambda(\|U(x)\|_\alpha, Mv^{1/\beta})\frac{\|U(x)\|_\alpha^{\lambda_1}v^{\frac{j_0}{\beta}-1}}{(Mv^{\frac{1}{\beta}})^{j_0-\lambda_2}}dv \\
&\xlongequal{t=\frac{Mv^{1/\beta}}{\|U(x)\|_\alpha}} \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\int_0^\infty k_\lambda(1,t)t^{\lambda_2-1}dt \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\int_0^\infty k_\lambda(v,1)v^{\lambda_1-1}dv.
\end{aligned}$$

因此, 式 (2.14) 成立. 证毕.

注 2.8 如果 $U_k(\infty)=\infty$ $(k=1,\dots,i_0)$, 则有 $w(\lambda_1,n)=K_\alpha(\lambda_1)$ $(n\in\mathbb{N}^{j_0})$.

3 主要结果及应用

对 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, 我们置函数

$$\Phi(x):=\frac{\|U(x)\|_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0}\mu_k(x))^{p-1}}\quad(x\in\mathbb{R}_+^{i_0}),\quad \Psi(n):=\frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0}v_n^{(l)})^{q-1}}\quad(n\in\mathbb{N}^{j_0}),$$

以及如下赋范空间

$$L_{p,\Phi}(\mathbb{R}_+^{i_0}):=\left\{f=f(x);\|f\|_{p,\Phi}:=\left(\int_{\mathbb{R}_+^{i_0}}\Phi(x)|f(x)|^pdx\right)^{\frac{1}{p}}<\infty\right\},$$

$$l_{q,\Psi}:=\left\{b=\{b_n\};\|b\|_{q,\Psi}:=\left(\sum_n\Psi(n)|b_n|^q\right)^{\frac{1}{q}}<\infty\right\},$$

$$L_{q,\Phi^{1-q}}(\mathbb{R}_+^{i_0}):=\left\{g=g(x);\|g\|_{q,\Phi^{1-q}}:=\left(\int_{\mathbb{R}_+^{i_0}}\Phi^{1-q}(x)|g(x)|^qdx\right)^{\frac{1}{q}}<\infty\right\},$$

$$l_{p,\Psi^{1-p}}:=\left\{c=\{c_n\};\|c\|_{p,\Psi^{1-p}}:=\left(\sum_n\Psi^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}}<\infty\right\},$$

我们有

定理 3.1 在定义 2.4 条件下, 如果 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), b_n \geq 0$, $f=f(x)\in L_{p,\Phi}(\mathbb{R}_+^{i_0})$, $b=\{b_n\}\in l_{q,\Psi}$, $\|f\|_{p,\Phi}$, $\|b\|_{q,\Psi}>0$, 则有如下等价不等式

$$\begin{aligned} I &:= \sum_n b_n \int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx \\ &< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}\|b\|_{q,\Psi}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J_1 &:= \left\{ \sum_n \frac{\prod_{j=1}^{j_0} \upsilon_n^{(j)}}{\|V_n\|_\beta^{j_0-p\lambda_2}} \left[\int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx \right]^p \right\}^{\frac{1}{p}} \\ &< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J_2 &:= \left\{ \int_{\mathbb{R}_+^{i_0}} \frac{\prod_{i=1}^{i_0} \mu_i(x)}{\|U(x)\|_\alpha^{i_0-q\lambda_1}} \left[\sum_n k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) b_n \right]^q dx \right\}^{\frac{1}{q}} \\ &< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)\|b\|_{q,\Psi}, \end{aligned} \tag{3.3}$$

其中

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1).$$

证明 由带权的 Hölder 不等式 [8], 我们有

$$\begin{aligned} I &= \sum_n \int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) \\ &\quad \times \left[\frac{\|U(x)\|_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{\frac{1}{p}} f(x)}{\|V_n\|_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{i=1}^{i_0} \mu_k(x))^{\frac{1}{q}}} \right] \left[\frac{\|V_n\|_\beta^{\frac{j_0-\lambda_2}{p}} (\prod_{i=1}^{i_0} \mu_k(x))^{\frac{1}{q}} b_n}{\|U(x)\|_\alpha^{\frac{i_0-\lambda_1}{q}} (\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{\frac{1}{p}}} \right] dx \\ &\leq \left[\int_{\mathbb{R}_+^{i_0}} W(\lambda_2, x) \frac{\|U(x)\|_\alpha^{p(i_0-\lambda_1)-i_0} f^p(x)}{(\prod_{i=1}^{i_0} \mu_i(x))^{p-1}} dx \right]^{\frac{1}{p}} \left[\sum_n w(\lambda_1, n) \frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})} \right]^{\frac{1}{q}}. \end{aligned}$$

因此, 我们由 (2.13) 和 (2.14) 式, 得到 (3.1) 式. 设

$$b_n := \frac{\prod_{j=1}^{j_0} \upsilon_n^{(j)}}{\|V_n\|_\beta^{j_0-p\lambda_2}} \left[\int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx \right]^{p-1}, \quad n\in\mathbb{N}^{j_0},$$

则有 $J_1=\|b\|_{q,\Psi}^{q-1}$. 因式 (3.2) 的右边式子是有限数, 故 $J_1<\infty$. 如果 $J_1=0$, 则式 (3.2) 成立; 如果 $J_1>0$, 则由式 (3.1), 有

$$\begin{aligned} \|b\|_{q,\Psi}^q &= J_1^p = I < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}\|b\|_{q,\Psi}, \\ \|b\|_{q,\Psi}^{q-1} &= J_1 < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}, \end{aligned}$$

即式 (3.2) 成立.

反之, 设式 (3.2) 成立, 由 Hölder 不等式 [8], 有

$$I = \sum_n \left[\frac{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{1/p}}{\|V_n\|_\beta^{(j_0/p)-\lambda_2}} \int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx \right] \frac{\|V_n\|_\beta^{(j_0/p)-\lambda_2}}{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{1/p}} b_n \leq J_1\|b\|_{q,\Psi}. \tag{3.4}$$

由式 (3.2) 知式 (3.1) 成立. 式 (3.1) 与式 (3.2) 等价.

同样的方法, 我们可以证明

$$I \leq \|f\|_{p,\Phi} J_2, \tag{3.5}$$

且式 (3.1) 等价于式 (3.3). 因此, 式 (3.1), (3.2) 和 (3.3) 是等价的. 证毕.

定理 3.2 在定理 3.1 条件下, 如果 $\upsilon_n^{(j)} \geq \upsilon_{n+1}^{(j)}$ $(n \in \mathbb{N})$, $U_i(\infty) = V_\infty^{(j)} = \infty$ $(i = 1, \ldots, i_0, j = 1, \ldots, j_0)$, 则在式 (3.1)–(3.3) 中的常数因子 $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ 是最佳的.

证明 对 $\varepsilon > 0$, 我们置

$$\widetilde{f} = \widetilde{f}(x), \quad \widetilde{f}(x) := \begin{cases} 0, & x \in \mathbb{R}_+^{i_0} \backslash [1,\infty)^{i_0}, \\ \|U(x)\|_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}} \displaystyle\prod_{i=1}^{i_0} \mu_i(x), & x \in [1,\infty)^{i_0}, \end{cases}$$

$$\widetilde{b} = \{\widetilde{b}_n\}, \quad \widetilde{b}_n := \|V_n\|_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} \prod_{j=1}^{j_0} \upsilon_n^{(j)} \quad (n \in \mathbb{N}^{j_0}).$$

由式 (2.8) 和 (2.9), 可得

$$\begin{aligned} \|\widetilde{f}\|_{p,\Phi} \|\widetilde{b}\|_{q,\Psi} &= \left[\int_{\mathbb{R}_+^{i_0}} \frac{\|U(x)\|_\alpha^{p(i_0-\lambda_1)-i_0} \widetilde{f}^p(x)}{(\prod_{i=1}^{i_0} \mu_i(x))^{p-1}} dx \right]^{\frac{1}{p}} \left[\sum_n \frac{\|V_n\|_\beta^{q(j_0-\lambda_2)-j_0} \widetilde{b}_n^q}{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{q-1}} \right]^{\frac{1}{q}} \\ &= \left(\int_{[1,\infty)^{i_0}} \|U(x)\|_\alpha^{-i_0-\varepsilon} \prod_{i=1}^{i_0} \mu_i(x) dx \right)^{\frac{1}{p}} \left(\sum_n \|V_n\|_\beta^{-j_0-\varepsilon} \prod_{j=1}^{j_0} \upsilon_n^{(j)} \right)^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon} \left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}}. \end{aligned}$$

由式 (2.12), 注意到 $\upsilon_n^{(j)} \geq \upsilon_{n+1}^{(j)} = \upsilon_j(y)$ $(n_j < y_j < n_j + 1)$, 有

$$\begin{aligned} \widetilde{I} &:= \sum_n \int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) \widetilde{f}(x) \widetilde{b}_n dx \\ &= \sum_n \int_{[1,\infty)^{i_0}} \int_{\{y \in \mathbb{R}_+^{j_0};\, n_j \leq y_j < n_j+1\}} \frac{k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta)}{\|U(x)\|_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} \|V_n\|_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \prod_{j=1}^{j_0} \upsilon_j(y) \prod_{i=1}^{i_0} \mu_k(x) dy dx \\ &> \int_{[1,\infty)^{i_0}} \sum_n \int_{\{y \in \mathbb{R}_+^{j_0};\, n_j \leq y_j < n_j+1\}} \frac{k_\lambda(\|U(x)\|_\alpha, \|V(y)\|_\beta)}{\|U(x)\|_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} \|V(y)\|_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \prod_{l=1}^{j_0} \upsilon_l(y) \prod_{k=1}^{i_0} \mu_k(x) dy dx \\ &= \int_{[1,\infty)^{i_0}} \int_{[1,\infty)^{j_0}} \frac{k_\lambda(\|U(x)\|_\alpha, \|V(y)\|_\beta) \prod_{l=1}^{j_0} \upsilon_l(y)}{\|U(x)\|_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} \|V(y)\|_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \prod_{k=1}^{i_0} \mu_k(x) dy dx. \end{aligned}$$

设 $u = U(x)$, $v = V(y)$, $c := \max_{1 \leq i \leq i_0, 1 \leq j \leq j_0} \{\mu_1^{(i)}, \upsilon_1^{(j)}\}$, 因 $U_\infty^{(k)} = V_\infty^{(l)} = \infty$, 我们有

$$\begin{aligned} \widetilde{I} &> \int_{[c,\infty)^{i_0}} \int_{[c,\infty)^{j_0}} \frac{k_\lambda(\|u\|_\alpha, \|v\|_\beta)}{\|u\|_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} \|v\|_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} dv du \\ &= \int_{[c,\infty)^{i_0}} \int_{[c,\infty)^{j_0}} \frac{k_\lambda(M_1[\sum_{i=1}^{i_0} (\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}, M_2[\sum_{j=1}^{j_0} (\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}) dy dx}{\{M_1[\sum_{i=1}^{i_0} (\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}\}^{i_0-\lambda_1+\frac{\varepsilon}{p}} \{M_2[\sum_{j=1}^{j_0} (\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}\}^{j_0-\lambda_2+\frac{\varepsilon}{q}}}. \end{aligned}$$

对

$$M_1 > ci_0^{1/\alpha}, \quad M_2 > cj_0^{1/\beta},$$

我们置

$$\Psi_1(u) = \begin{cases} 0, \quad 0 < u \le \dfrac{c^\alpha i_0}{M_1^\alpha}, \\ k_\lambda\left(M_1 u^{1/\alpha}, M_2\left[\displaystyle\sum_{j=1}^{j_0}\left(\frac{y_j}{M_2}\right)^\beta\right]^{\frac{1}{\beta}}\right)\dfrac{1}{(M_1 u^{1/\alpha})^{i_0-\lambda_1}}, \quad \dfrac{c^\alpha i_0}{M_1^\alpha} < u \le 1, \end{cases}$$

$$\Psi_2(v) = \begin{cases} 0, \quad 0 < v \le \dfrac{c^\beta j_0}{M_2^\beta}, \\ k_\lambda(M_1 u^{1/\alpha}, M_2 v^{1/\beta})\dfrac{1}{(M_2 v^{1/\beta})^{j_0-\lambda_2}}, \quad \dfrac{c^\beta j_0}{M_2^\beta} < v \le 1. \end{cases}$$

再次运用式 (2.7), 有

$$\begin{aligned} \widetilde{I} > & \lim_{M_1\to\infty}\lim_{M_2\to\infty}\frac{M_1^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\frac{M_2^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0}\Gamma(\frac{j_0}{\beta})}\int_{c^\alpha i_0/M_1^\alpha}^1 u^{\frac{i_0}{\alpha}-1} \\ & \times\left[\int_{c^\beta j_0/M_2^\beta}^1\frac{k_\lambda(M_1 u^{\frac{1}{\alpha}}, M_2 v^{\frac{1}{\beta}})v^{\frac{j_0}{\beta}-1}}{(M_1 u^{\frac{1}{\alpha}})^{i_0-\lambda_1+\frac{\varepsilon}{p}}(M_2 v^{\frac{1}{\beta}})^{j_0-\lambda_2+\frac{\varepsilon}{q}}}dv\right]du. \end{aligned}$$

在上述积分中作变换

$$x = M_1 u^{\frac{1}{\alpha}}, \quad y = M_2 v^{\frac{1}{\beta}},$$

得

$$\begin{aligned} \widetilde{I} > & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\int_{ci_0^{1/\alpha}}^\infty x^{\lambda_1-\frac{\varepsilon}{p}-1}\left(\int_{cj_0^{1/\beta}}^\infty k_\lambda(x,y)y^{\lambda_2-\frac{\varepsilon}{q}-1}dy\right)dx \\ = & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_0^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx \\ = & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right. \\ & \left.+\int_{ci_0^{1/\alpha}}^\infty x^{-\varepsilon-1}\left(\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right] \\ = & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\frac{1}{\varepsilon(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\ & \left.+\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty\left(\int_{cj_0^{1/\beta}v}^\infty x^{-\varepsilon-1}dx\right)k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right] \\ = & \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\times\left[\frac{1}{(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\ & \left.+\frac{1}{(cj_0^{1/\beta})^\varepsilon}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right]. \end{aligned}$$

如果存在常数 $K \le K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$, 使得用 K 替换 $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ 之后, 式 (3.1) 仍然成立, 则有 $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Phi}\|\widetilde{b}\|_{q,\Psi}$, 亦即

$$\begin{aligned}&\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\\&\quad\times\left[\frac{1}{(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}}k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv+\frac{1}{(cj_0^{1/\beta})^\varepsilon}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right]\\&\quad<K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon O(1)\right)^{\frac{1}{q}}.\end{aligned}$$

令 $\varepsilon\to 0^+$, 由 Fatou 引理 [9] 有

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1)\le K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

推出

$$K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\le K.$$

因此, $K=K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ 是式 (3.1) 的最佳常数因子. 式 (3.2) (式 (3.3)) 中的常数因子也是最佳的, 否则, 由式 (3.4) (式 (3.5)) , 我们会得出式 (3.1) 的常数因子不是最佳值的矛盾. 证毕.

特别地, 在定理 3.1–3.2 中, 取 $\mu_i(t)=1\,(i=1,\dots,i_0)$, $v_j^{(l)}=1\,(l=1,\dots,j_0;\ j=1,\dots,n)$, 置

$$\varphi(x):=\|x\|_\alpha^{p(i_0-\lambda_1)-i_0}\quad(x\in\mathbf{R}_+^{i_0}),\quad\psi(n):=\|n\|_\beta^{q(j_0-\lambda_2)-j_0}\quad(n\in\mathbb{N}^{j_0}),$$

我们有

推论 3.3 在定义 2.4 条件下, 设 $p>1, \frac{1}{p}+\frac{1}{q}=1$, $f(x), b_n\ge 0$, $f=f(x)\in L_{p,\varphi}(\mathbb{R}_+^{i_0})$, $b=\{b_n\}\in l_{q,\psi}$, $\|f\|_{p,\varphi}$, $\|b\|_{q,\psi}>0$, 则有如下等价不等式:

$$\sum_n b_n\int_{\mathbb{R}_+^{i_0}}k_\lambda(\|x\|_\alpha,\|n\|_\beta)f(x)dx<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\varphi}\|b\|_{q,\psi},\tag{3.6}$$

$$\left\{\sum_n\frac{1}{\|n\|_\beta^{j_0-p\lambda_2}}\left[\int_{\mathbb{R}_+^{i_0}}k_\lambda(\|x\|_\alpha,\|n\|_\beta)f(x)dx\right]^p\right\}^{\frac{1}{p}}<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\varphi},\tag{3.7}$$

$$\left\{\int_{\mathbb{R}_+^{i_0}}\frac{1}{\|x\|_\alpha^{i_0-q\lambda_1}}\left[\sum_n k_\lambda(\|x\|_\alpha,\|n\|_\beta)b_n\right]^q dx\right\}^{\frac{1}{q}}<K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|b\|_{q,\psi},\tag{3.8}$$

其中, 常数因子

$$K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)=\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}k(\lambda_1)$$

是最佳值.

推论 3.4 在定义 2.4 条件下 (设 $i_0=j_0=1$), 置

$$\Phi_1(x):=\frac{(U_1(x))^{p(1-\lambda_1)-1}}{(\mu_1(x))^{p-1}},$$

$$\Psi_1(n):=\frac{(V_n^{(1)})^{q(1-\lambda_2)-1}}{(v_n^{(1)})^{q-1}}\quad(x\in\mathbb{R}_+,\ n\in\mathbb{N}),$$

如果 $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), b_n \geq 0$, $f=f(x)\in L_{p,\Phi_1}(\mathbb{R}_+)$, $b=\{b_n\}\in l_{q,\Psi_1}$, $\|f\|_{p,\Phi_1}, \|b\|_{q,\Psi_1}>0$, 则有如下等价不等式:

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} k_\lambda(U_1(x), V_n^{(1)}) f(x)dx < k(\lambda_1)\|f\|_{p,\Phi_1}\|b\|_{q,\Psi_1}, \tag{3.9}$$

$$\left\{\sum_{n=1}^{\infty} \frac{\upsilon_n^{(1)}}{(V_n^{(1)})^{1-p\lambda_2}}\left[\int_0^{\infty} k_\lambda(U_1(x), V_n^{(1)}) f(x)dx\right]^p\right\}^{\frac{1}{p}} < k(\lambda_1)\|f\|_{p,\Phi_1}, \tag{3.10}$$

$$\left\{\int_0^{\infty} \frac{\mu_1(x)}{(U_1(x))^{1-q\lambda_1}}\left[\sum_{n=1}^{\infty} k_\lambda(U_1(x), V_n^{(1)}) b_n\right]^q dx\right\}^{\frac{1}{q}} < k(\lambda_1)\|b\|_{q,\Psi_1}. \tag{3.11}$$

此外, 如果还满足条件 $\upsilon_n^{(1)} \geq \upsilon_{n+1}^{(1)}$ $(n\in\mathbb{N})$, $U_1(\infty)=V_\infty^{(1)}=\infty$, 则常数因子 $k(\lambda_1)$是最佳值.

注 3.5 取 $i_0=j_0=1$ $(\mu_1(t)=1, \upsilon_j^{(1)}=1$ $(j=1,\ldots,n))$, 则式 (3.6) (式 (3.9)) 变为 (1.6). 因此, 式 (3.6) (式 (3.9)) 是 (1.6) 的推广. 式 (3.1) 也是如此.

4 算子表示

在定理 3.2 条件下, 置

$$c_n := \frac{\prod_{j=1}^{j_0}\upsilon_n^{(j)}}{\|V_n\|_\beta^{j_0-p\lambda_2}}\left[\int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx\right]^{p-1}, \quad n\in\mathbb{N}^{j_0},$$

$$c=\{c_n\}, \quad \|c\|_{p,\Psi^{1-p}} = J_1 < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}<\infty,$$

我们给出以下定义:

定义 4.1 定义一个半离散多重 Hardy–Hilbert 型算子 $T_1: L_{p,\Phi}(\mathbb{R}_+^{i_0}) \to l_{p,\Psi^{1-p}}$ 如下: 对任意的 $f\in L_{p,\Phi}(\mathbf{R}_+^{i_0})$, 存在唯一的表示 $T_1 f = c \in l_{p,\Psi^{1-p}}$, 使得

$$T_1 f(n) := \int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx \quad (n\in\mathbb{N}^{j_0}). \tag{4.1}$$

对 $b\in l_{q,\Psi}$, 我们定义 $T_1 f$ 和 b 的形式内积如下:

$$(T_1 f, b) := \sum_n \left[\int_{\mathbb{R}_+^{i_0}} k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) f(x)dx\right] b_n. \tag{4.2}$$

那么由式 (3.1) 和 (3.2), 我们有如下等价不等式:

$$(T_1 f, b) < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{4.3}$$

$$\|T_1 f\|_{p,\Psi^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)\|f\|_{p,\Phi}, \tag{4.4}$$

则算子 T_1 有界且满足

$$\|T_1\| := \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbb{R}_+^{i_0})} \frac{\|T_1 f\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi}} \leq K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{4.5}$$

由定理 3.2, 式 (4.4) 中的常数因子 $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ 是最佳的, 我们得到

$$\|T_1\| = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1). \tag{4.6}$$

在定理 3.2 条件下, 置

$$g(x) := \frac{\prod_{i=1}^{i_0} \mu_i(x)}{\|U(x)\|_\alpha^{i_0 - q\lambda_1}} \left[\sum_n k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) b_n \right]^{q-1}, \quad x \in \mathbb{R}_+^{i_0},$$

$$g = g(x), \quad \|g\|_{q,\Phi^{1-q}} = J_2 < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|b\|_{q,\Psi} < \infty,$$

我们给出以下定义:

定义 4.2 定义一个半离散多重 Hardy–Hilbert 型算子 $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(\mathbb{R}_+^{i_0})$ 如下: 对任意的 $b \in l_{q,\Psi}$, 存在唯一的表示 $T_2 b = g \in L_{q,\Phi^{1-q}}(\mathbb{R}_+^{i_0})$, 使得

$$T_2 b(x) := \sum_n k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) b_n \quad (x \in \mathbb{R}_+^{i_0}). \tag{4.7}$$

对 $f \in L_{p,\Phi}(\mathbb{R}_+^{i_0})$, 我们定义 $T_2 b$ 和 f 的形式内积如下:

$$(f, T_2 b) := \int_{\mathbb{R}_+^{i_0}} f(x) \left[\sum_n k_\lambda(\|U(x)\|_\alpha, \|V_n\|_\beta) b_n \right] dx. \tag{4.8}$$

则由式 (3.1) 和 (3.3), 有如下的等价不等式:

$$(f, T_2 b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|f\|_{p,\Phi} \|b\|_{q,\Psi}, \tag{4.9}$$

$$\|T_2 b\|_{q,\Phi^{1-q}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \|b\|_{q,\Psi}. \tag{4.10}$$

可见 T_2 是有界算子, 满足

$$\|T_2\| := \sup_{b(\neq\theta) \in l_{q,\Psi}} \frac{\|T_2 b\|_{q,\Phi^{1-q}}}{\|b\|_{q,\Psi}} \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{4.11}$$

由定理 3.2, 式 (4.10) 的常数因子 $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ 是最佳的, 因此

$$\|T_2\| = K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1). \tag{4.12}$$

例 4.3 (i) 在例 2.5 中, 对 $k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}$, 由式 (4.6) 和 (4.12), 我们有

$$\|T_1\| = \|T_2\| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\lambda + 2\eta}{(\lambda_1 + \eta)(\lambda_2 + \eta)}.$$

(ii) 对 $k_\lambda(x,y) = \frac{1}{x^\lambda + y^\lambda}$ ($\lambda_1 > 0$, $0 < \lambda_2 \leq j_0$, $\lambda_1 + \lambda_2 = \lambda$), 我们有

$$k(\lambda_1) = \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})},$$

则由式 (4.6) 和 (4.12), 可得

$$\|T_1\| = \|T_2\| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}.$$

(iii) 对 $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ ($\lambda_1 > 0$, $0 < \lambda_2 \leq j_0$, $\lambda_1 + \lambda_2 = \lambda$), 我们有

$$k(\lambda_1) = \left[\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \right]^2,$$

则由式 (4.6) 和 (4.12), 可得

$$\|T_1\| = \|T_2\| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2.$$

注 4.4 取 $0 < \lambda_1 + \eta \le i_0$, $0 < \lambda_2 + \eta \le j_0$,

$$k_\lambda(x,y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}} \quad (x,y>0),$$

则式 (3.1) 变为文 [41, 式 (23)], 它是式 (1.4) (取 $\mu_i = \upsilon_j = 1$ $(i,j\in\mathbb{N})$) 的推广.

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